# Differential Geometry

**Instructor:** Jianfeng Lin

**Notes Taker:** Zejin Lin

TSINGHUA UNIVERSITY.

linzj23@mails.tsinghua.edu.cn

lzjmaths.github.io

February 19, 2025

## Contents

# 1 Smooth Manifold

**Definition 1.1** (Topological manifold)**.** A space $M$ is called a topological manifold if

1. locally Euclidean
2. Hausdorff
3. second countable

**Definition 1.2** (Smooth Manifold)**.** A smooth structure is given by an equivalence class of smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ *s.t.* $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth $\forall \alpha, \beta$. $M = \cup U_\alpha$.

A **smooth manifold** is a topological manifold with a smooth structure.

Define when a continuous map $f : M_1 \to M_2$ is smooth if $\forall (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2$, we have $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth.

**Definition 1.3.** Given $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$. A homeomorphism $f : M_1 \to M_2$ is called a diffeomorphism if $f$, $f^{-1}$ is smooth.

In this case we say $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ are diffeomorphism.

**Theorem 1.4** (Kervaire)**.** *$\exists$ 1 10-dimensional topological manifold without smooth manifold.*

**Theorem 1.5** (Milnor)**.** *$\exists$ a smooth manifold $M$ s.t. $M \cong S^7$ but not in diffeomorphism meaning.*

**Theorem 1.6** (Kervaire-Milnor)**.** *$\exists$ 28 smooth structures (up to orientation preserving diffeomorphism) on $S^7$*

**Theorem 1.7** (Morse-Birg)**.** *On $S^7$. If $n \leqslant 3$, then any $n$-dimensional topological manifold $M$ has a unique smooth structure up to diffeomorphism.*

**Theorem 1.8** (Stallings)**.** *If $n \neq 4$, then $\exists$ a unique smooth structure on $\mathbb{R}^n$ up to diffeomorphism.*

**Theorem 1.9** (Donaldson-Freedom-Gompf-Faubes)**.** *$\exists$ uncountable smooth structures on $\mathbb{R}^4$ up to diffeomorphism.*

**Definition 1.10** (topological manifold with boundary)**.** A space $M$ is called a topological manifold with boundary if

1. $M$ is Hausdorff
2. $M$ is second countable
3. $\forall p \in M$, $\exists$ a neighbourhood $U$ of $p$ and a homeomorphism $\varphi : U \to V$ where $V$ is open in $\mathbb{H}^n$

We say a manifold $M$ is closed if $M$ is compact and $\partial M$ is empty.

Our motivation for studying manifold is to study the space of solution for equations.

**Question 1.** Given $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $q \in \mathbb{R}^n$, when is $f^{-1}(q)$ is a smooth manifold?

For $f : U \to \mathbb{R}^n$ smooth, $U$ open in $\mathbb{R}^m$, the differential of $f$ at $p \in U$ denoted as $\mathrm{d}f(p)$.

**Definition 1.11.** We say $p \in U$ is a **regular point** of $f$ if $\mathrm{d}f(p)$ is surjective. Otherwise we say $p \in U$ is a **critical point**.
A point $q \in \mathbb{R}^n$ is called a **regular value** of $f$ if $\forall p \in f^{-1}(q)$ , $p$ is a regular point of $f$.
A point $q \in \mathbb{R}^n$ is called a **critical value** of $f$ if $\forall p \in f^{-1}(q)$ , $p$ is a critical point of $f$.

**Theorem 1.12** (Implicit function theorem)**.** *If $p \in U$ is a regular point of $f : U \to \mathbb{R}^n$. Then there exists*

- *An open neighbourhood $V$ of $p$ in $U$*
- *An open subset $V'$ of $\mathbb{R}^m$*
- *A diffeomorphism $\varphi : V \to V'$ such that $P \circ \varphi = f$ where $P$ is the projection from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

*In other words, near a regular point, we can do local coordinate change to turn $f$ into the projection.*

**Remark 1.13.** Inverse function theorem and Implicit function theorem gives a way to find the related from "a point" to "a beibourhood"!

In particular, we have a homeomorphism

$$f^{-1}(f(p)) \cap V \xrightarrow[\text{restriction of } \varphi]{\cong} \{(x_1, \ldots, x_m) \in V' | (x_1, \cdots .x_n) = f(p)\}$$

*i.e.* if we set $M = f^{-1}(f(p))$, then $(M \cap V, \varphi_p)$ is a chart that contains $p$.

**Corollary 1.14.** *If $q$ is a regular value of $f : U \to \mathbb{R}^n$ then $f^{-1}(q)$ is a smooth manifold.*

**Remark 1.15.** It suffices to show that the corresponding charts are compatible.

**Theorem 1.16** (Sard)**.** *If $f : U \to \mathbb{R}^n$ is a smooth map, then the set of critical values of $f$ has measure $0$.*

**Remark 1.17.** For a "generic" $q$, $f^{-1}(q)$ is a manifold of dimension $m - n$.

**Corollary 1.18.** *If $f : U \to \mathbb{R}^n$ is smooth and $m < n$ then $f(U)$ has measure $0$.*

## 1.1 Lie Groups and Homogeneous Spaces

**Definition 1.19.** We say $G$ is a **Lie group** if it is a topological group with a smooth structure such that the multiplication map $\cdot : G \times G \to G$ and the inverse map $G \rightsquigarrow G$ is smooth.

**Example 1.20.** $GL(n, \mathbb{R}) = \{n \times n$ matrices with non-zero determinant$\} \subset \mathbb{R}^{n \times n}$
$O(n) = \{A \in GL(n, \mathbb{R}) | AA^T = I\}$
$SO(n) = \{A \in O(n) | \det A = 1\}$
$U(n) = \{A \in GL(n, \mathbb{C}) | A\overline{A}^T = I\}$
$SU(n) = \{A \in U(n) | \det A = 1\}$

**Exercise 1.21.**

$$O(1) \cong S^2 \qquad SO(1) \cong * \tag{1.1}$$

$$SO(2) \cong S^1 \qquad SO(3) \cong \mathbb{RP}^3 \tag{1.2}$$

$$SU(2) \cong S^3 \qquad U(n) \cong S^1 \times SU(n) \tag{1.3}$$

The last one is a diffeomorphism but do not preserve the multiplicatioin, *i.e.* not an isomorphism of Lie group.

**Theorem 1.22** (Carton)**.** *Let $H$ be a closed subgroup of Lie group $G$. Then $H$ is a Lie group. More precisely, $H$ is topological manifold, carries a canonical smooth structure that makes the multiplication and inverse smooth. Also, $G/H$ is a smooth manifold*

**Definition 1.23.** Let $M$ be a smooth manifold. We say $M$ is a **homogeneous space** if $\exists$ a Lie group $G$ with a smooth transitive action $\rho : G \times M \to M$.

**Definition 1.24.** For $M$ be a homogeneous space. The **isotropy** group of $x \in M$ is defined as

$$Iso(x) = \{g \in G | gx = x\}$$

closed subgroup of $G$

Given any $x, x' \in M$, $Iso(x) \cong Iso(x')$ because the group action is transitive.

Hence, we have a well-defined map

$$p : G/_{Iso(x)} \to M \tag{1.4}$$

$$g \mapsto gx \tag{1.5}$$

**Theorem 1.25.** *$p$ is always a diffeomorphism.*

Therefore, we have this proposition

**Proposition 1.26.** *$M$ is a homogeneous space $\Leftrightarrow$ $M = G/H$ for some closed subgroup $H$.*

**Example 1.27.** If $M = S^n$, let $G = SO(n+1)$.

Then $Iso(1, 0, \cdots, 0) \cong SO(n)$.

So $S^n \cong SO(n+1)/(SO(n))$.

Similarly, we can prove $\mathbb{RP}^n \cong SO(n+1)/(O(n))$, $\mathbb{CP}^n \cong SO(n+1)/(U(n))$

The isotropy k dimensional linear subspaces of $\mathbb{R}^n$ can be $O(k) \times O(n-k)$ if $G = O(n)$

A connected closed surface is a homogeneous space if and only if it is diffeomorphic to $\mathbb{RP}^2, S^2, T^2$ and Klein bottle.

**Theorem 1.28** (Whithead)**.** *Any smooth manifold has a triangulation.*

**Theorem 1.29** (Poincare-Hopf)**.** *$G$ is compact Lie group $\Rightarrow \chi(G) = 0$.*

**Theorem 1.30** (Mostow2005)**.** *$M$ is a compact homogeneous space $\Rightarrow \chi(M) \geqslant 0$.*

## 1.2 Bump Function and Partition of Unity

**Theorem 1.31** (Urysohn smooth version)**.** *Given $M$, closed disjoint $A, B$, $\exists$ smooth $f : M \to [0,1]$ s.t. $f|_A = 0, f|_B = 1$.*

**Theorem 1.32** (Tietze)**.** *Given $M$, closed $A$, smooth $f : A \to \mathbb{R}^n$, there exists smooth $\widehat{f} : M \to \mathbb{R}^n$ s.t. $\widehat{f}|_A = f$*

To prove these and much more result we need partition of unity theorem.

First we define bump function.

**Lemma 1.33.** *Let $U$ be a neighbourhood of $p \in M$. Then $\exists$ smooth $\sigma : M \to [0,1]$ s.t.*

1. *$\sigma \equiv 1$ near $p$*
2. *Supp $\sigma \subset U$*

*Such $\sigma$ is called a **bump function** at $p$, supported in $U$.*

**Definition 1.34.** An open cover of a space $X$ is **locally finite** if any point has a neighbourhood that intersects only finite many open sets of this cover.

**Proposition 1.35.** *Given compact $K \subset U$ and open neighbourhood $U$ of $K$, $\exists$ a smooth $g : M \to [0,+\infty)$ s.t. $g|_K \equiv 1$ and Supp $g \subset U$.*

**Definition 1.36.** An **exhaust** of a space $X$ is a sequence of open sets $\{U_i\}$ *s.t.*

1. $X = \bigcup_{i=1}^{\infty} U_i$
2. $\overline{U_i}$ is compact and contained in $U_{i+1}$

**Theorem 1.37.** *Any topological manifold has an exhaust.*

Given two open covers $\mathcal{U}, \mathcal{V}$, we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if $\forall U_\alpha \in U$, $\exists V_\beta \in \mathcal{V}$ *s.t.* $V_\beta \subset U_\alpha$.

We say a space $X$ is paracompact if any open cover has a locally finite refinement.

Actually, any metric space is paracompact.(The proof is hard)

**Proposition 1.38.** *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of a topological manifold $M$. Then there exists countable open covers $\mathcal{W} = \{W_i\}$, $\mathcal{V} = \{V_i\}$ s.t.*

- *For any $i$, $\overline{V_i}$ is compact and $\overline{V_i} \subset W_i$*
- *$\mathcal{W}$ is locally finite.*
- *$\mathcal{W}$ is a refinement of $\mathcal{U}$.*

As a corollary, we have any topological manifold is paracompact.

**Definition 1.39.** Given open cover $\mathcal{U}$ of a smooth $M$, a partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in \mathcal{A}}$ *s.t.*

1. $\forall p \in M$, $\exists$ only finitely many $\alpha \in \mathcal{A}$ *s.t.* $p \in Supp\, \rho_\alpha$
2. $\sum\limits_{\alpha \in \mathcal{A}} \rho_\alpha(p) = 1$
3. $Supp\, \rho_\alpha \subset U_\alpha$

**Theorem 1.40** (Existence of P.O.U)**.** *For any open cover $\mathcal{U}$ of smooth $M$, $\exists$ a P.O.U subordinate to $\mathcal{U}$*

**Theorem 1.41** (Whitney approximation theorem)**.** *Given any smooth $M$, any closed $A$ and any continuous $f : M \to \mathbb{R}$, $\delta : M \to (0, +\infty)$. Suppose $f$ is smooth on $A$. Then $\exists g : M \to \mathbb{R}$ smooth s.t.*

- $g|_A = f|_A$
- $\forall p \in M$, $|g(p) - f(p)| < \delta(p)$.

# 2 Tangent space and tangent vectors

## 2.1 Tangent Space

Given $p \in M$, consider the set $C_p^\infty(M) = \{\text{smooth function } V \to \mathbb{R}\}/_\sim$ where $f_1 \sim f_2$ if and only if $\exists$ neighbourhood $U$ of $p$, $f_1|_U = f_2|_U$.

$C_p^\infty(M)$ is the space of **genus of smooth function** near $p$.

A **partial-derivative** of $p$ is a $\mathbb{R}$-linear map $D : C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

**Definition 2.1.** A **tangent vector** of $M$ at $p$ is a partial-derivative at $p$.

Define the **tangent space** $T_pM = \{\text{all partial-derivative at } p\}$, which is a $\mathbb{R}$-vector space.

**Proposition 2.2.** *For $M = U \subset \mathbb{R}^n$ open. We have $\{\frac{\partial}{\partial x_i}\}$ is a basis for $T_pU$.*

**Proposition 2.3.**

$$\frac{\partial}{\partial x^i}|_p = \sum_{1 \leqslant i \leqslant n} \frac{\partial y^i}{\partial x^i} \cdot \frac{\partial}{\partial y^i}|_p$$

Now we try to define differential of a smooth map.

$M, N$ smooth manifolds, $C^\infty(N, M) = \{\text{smooth } F : N \to M\}$.

Given $F \in C^\infty(N, M)$, $F$ induces $F^* : C_{F(p)}^\infty(M) \to C_p^\infty(N)$, $f \mapsto f \circ F$.

By taking dual, we get

$$F_* : T_pN \to T_{F(p)}M$$

we also write $F_*$ as $F_{*,p}$, call it the **differential** of $F$ at $p$.

where

$$F_*(\frac{\partial}{\partial x^i}|_p) = \sum_k \frac{\partial F^k}{\partial x^i} \cdot \frac{\partial}{\partial y^k}|_{F(p)}$$

**Proposition 2.4.** *The differential satisfies the composition law.*

$$(G \circ F)_* = G_* \circ F_* : T_pN \to T_{G\circ F(p)}W$$

**Definition 2.5.** A smooth **curve** is a smooth map $\gamma : (a, b) \to M$. We say $\gamma$ starts at $p$ if $\gamma(0) = p$. We define the **velocity** of $\gamma$ at $\gamma(0)$ as $\gamma_*(\frac{\partial}{\partial t}|_0) \in T_{\gamma(0)}M$

Take charts $(U, x^1, \cdots, x^n)$ about $p$, let $\gamma^i = x^i \circ \gamma$.

We say $\gamma, \delta$ are **tangent** to each other at $p$ if $(\gamma^i)'(0) = (\delta^i)'(0)$.

Now we can define

$$(T_pM)_{curve} := \{\text{smooth curves } \gamma \text{ starting at } p\}/_\sim$$

where $\gamma \sim \delta$ iff they are tangent to each other.

Then these definition is more geometric.

**Lemma 2.6.** *Given $F \in C^\infty(M, M)$, $p \in N$, the diagram commutes:*

$$\begin{array}{ccc} \gamma \in (T_pN)_{curve} & \xrightarrow{\cong} & T_pN \\ \big\downarrow & & \big\downarrow \\ F \circ \gamma \in (T_{F(p)}M)_{curve} & \xrightarrow{\cong} & T_{F(p)}M \end{array}$$

## 2.2 Tangent Bundle

Let $(M, \mathcal{A})$ be a smooth manifold, $TM = \bigcup_{p\in M} T_pM$, called the **tangent bundle**

Now we want to define a natural topology and smooth structure on $TM$. Take any chart $(U, \varphi) = (U, x^1, \cdots, x^n) \in \mathcal{A}$.

We have a map

$$\widehat{\varphi} : TU \xrightarrow{\cong} \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$$

$$X \in T_pU \mapsto (\varphi(p), X^1, \cdots, X^n)$$

where $X = \sum X^i \frac{\partial}{\partial x^i}|_p$.

Then pull back standard topology on $\varphi(U) \times \mathbb{R}^n$ to a topology on $TU$.

$$\mathcal{B} = \{\widehat{\varphi}^{-1}(V) | (\varphi, U) \in \mathcal{A}, V \text{ open in } \varphi(U) \times \mathbb{R}^n \}$$

There is some fact in topology:

- $\mathcal{B}$ is a basis
- $\mathcal{B}$ generates a Hausdorff, second countable topology on $TM$.

So $TM$ is a topological manifold covered by charts $\widehat{\mathcal{A}} = \{(TU, \widehat{\varphi}) | (U, \varphi) \in \mathcal{A}\}$.

Given $(TU, \widehat{\varphi}), (TV, \widehat{\psi}) \in \widehat{\mathcal{A}}$, the transition function is

$$\varphi(U \cap V) \times \mathbb{R}^n \xrightarrow{\widehat{\psi} \circ \widehat{\varphi}^{-1}} \psi(U \cap V) \times \mathbb{R}^n \tag{2.1}$$

$$(p, x) \mapsto (\psi \circ \varphi^{-1}, J(\psi \circ \varphi^{-1})|_p(X)) \tag{2.2}$$

So $\widehat{\mathcal{A}}$ is a smooth atlas on $TM$, making $TM$ into a smooth manifold.

**Definition 2.7** (vector bundle)**.** Given a continuous map $f : E \to B$, we say $f$ is a $n$-dimensional **vector bundle** if: $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $B$ and homeomorphisms $\{f^{-1}(U_\alpha) \underset{\cong}{\xrightarrow{\rho_\alpha}} U_\alpha \times \mathbb{R}\}$ *s.t.*

- $\begin{array}{ccc} f^{-1}(U_\alpha) & \xrightarrow{\rho_\alpha} & U_\alpha \times \mathbb{R}^n \\ \downarrow f & \swarrow \scriptstyle{projection} & \\ U_\alpha & & \end{array}$ commutes for $\alpha \in I$.
- $\forall\, p \in U_\alpha \cap U_\beta$, the map

$$\mathbb{R}^n = \{p\} \times \mathbb{R}^n \xrightarrow{\rho_\alpha} f^{-1}(p) \xrightarrow{\rho_\beta} \{p\} \times \mathbb{R}^n = \mathbb{R}^n$$

is linear.

Call $f^{-1}(p)$ the **fiber** over $p$.

**Proposition 2.8.** *Given vector bundle $f : E \to B$, the fiber $f^{-1}(p)$ has a structure of a vector space.*

**Example 2.9** (Product bundle)**.** $E = \mathbb{R}^n \times B$

**Example 2.10** (Tautological bundle)**.**

$$B = \mathbb{CP}^n = \{\text{1-dim complex subspace of } \mathbb{C}^{n+1}\}, E = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}\}$$

And we map $(L, v) \mapsto L$

Given vector bundles $E_1 \xrightarrow{\pi_1} B_1, E_2 \xrightarrow{\pi_2} B_2$, a bundle map consists of $(\hat{f}, f)$ *s.t.*

- $\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$ commutes.
- $\forall b \in B$, $\hat{f} : \pi_1^{-1}(b) \to \pi_2^{-1}(f(b))$ is linear.

If $\hat{f}, f$ are diffeomorphisms, then we call $(\hat{f}, f)$ an **isomorphism** of vector bundle.

An isomorphism to a product bundle is called a **trivialization**. An bundle is **trivial** if it has a trivialization.

**Example 2.11.** $TS^1, TS^2$ are both trivial.

$S^1 \cong O(1) \cong SO(2), S^3 \cong SU(2)$

**Theorem 2.12.** *If $G$ is a Lie group, then $TG$ is trivial.*

*Proof.* For $(x^1, x^2, \cdots, x^n)$ is a basis of $T_eG$ The bundle isomorphism is

$$G \times \mathbb{R}^n \xrightarrow{\varphi} TG,\ (g, c^1, \cdots, c^n) \mapsto (g, (l_g)_{*,e}(\sum_i c^i x^i))$$

where

$$l_g : G \to G, h \mapsto gh$$

is a diffeomorphism. Hence, it induces the isomorphism $(l_g)_*$

□

**Proposition 2.13** (Adams, 1960s). *$TS^n$ is trivial if and only if $n = 0, 1, 3, 7$.*

**Proposition 2.14.**

1. *Given any $F \in C^\infty(M, N)$, $F_* : TM \to TN$ is a bundle map.*

2. *$TS^n$ is isomorphic to the following bundle:*

$$B = s^n \qquad E = \{(p, v) \in S^n \times \mathbb{R}^{n+1} | v \perp p\}$$

**Definition 2.15** (smooth section). Given a smooth vector bundle $\pi : E \to B$, a **smooth section** is a smooth map $S : B \to E$ *s.t.* $\pi \circ S = id_b$.

$s_0 : B \to E, b \mapsto 0 \in$ 0-vector in $\pi^{-1}b$ .

## 2.3 Vector Field, Curves and Flows

**Definition 2.16.** A (tangent) **vector field** is a smooth section of $TM$. *i.e.* a smooth map $M \xrightarrow{X} TM$ *s.t.* $X(p) \in T_pM, \forall p \in M$

Given any $f : \mathbb{R}^n \to \mathbb{R}$, define the **gradient vector field**

$$\nabla f_p := \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}$$

**Example 2.17.** $X = f^1 \partial x^1 + f^2 \partial x^2$ is a gradient field if and only if $\dfrac{\partial f^1}{\partial x^2} = \dfrac{\partial f^2}{\partial x^1}$

**Theorem 2.18** (Poincare-Hopf). *For closed $M$, $M$ has a nowhere vanishing vector field if and only if $\chi(M) = 0$.*

So $S^n$ has a nowhere vanishing vector field if and only if $n$ is odd.

**Theorem 2.19** (MaoQiu). *$S^2$ has no no-where vanishing vector field.*

So We cannot smooth out all the hairs on a ball.

Given a vector field $X = \{X_p\}_{p\in M}$, a curve $\gamma : (a, b) \to M$ is called an **integral curve** of $X$ if $\gamma'(t) = X_{\gamma(t)}$, $\forall t \in (a, b)$, where $\gamma'(t) = \gamma_*(\frac{\partial}{\partial t}) \in T_{\gamma(t)}M$.

We say $\gamma$ is maximal if the domain cannot be extended to a larger interval.

Denote the set of all smooth vector fields on $M$ by $\mathfrak{X}M$

Recall that $\gamma$ is maximal if it's domain can not be extended to a large open interval.

In a local chart $(U, x^1, \cdots, x^n)$, $X|_U = \sum_{i=1}^{n} a^i \partial x^i$. Then $\gamma$ is an integral curve if and only if $(\gamma^i)'(t) = a^i(\gamma(t))$, $\forall 1 \leqslant i \leqslant n$, where $\gamma^i = x^i \circ \gamma : (a, b) \to \mathbb{R}$.

And in this case the initial value condition: $\gamma(0) = p \Leftrightarrow \gamma^i(0) = p^i$.

Locally, solving integral curve starting at $p$ is equivalent to solving ODE with initial value $p^1, \cdots, p^n$. By existence and uniqueness of solutions of ODE, we have

**Theorem 2.20** (Fundamental theorem of integral curve). *Let $X \in \mathfrak{X}M$, $p \in M$, then:*

*(1) (Uniqueness) Given any two integral curves $\gamma_1, \gamma_2 : (a, b) \to M$, then we have:*

$$\gamma_1(c) = \gamma_2(c) \textit{ for some } c \in (a, b) \ \Rightarrow \gamma_1 = \gamma_2$$

*(2) there exists a unique max integral curve $\gamma : (a(p), b(p)) \to M$ starting at $p$.*

*(3) (integral curve smoothly depend on initial values) $\exists$ Nbh $U$ of $p$, $\varepsilon > 0$, and smooth $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ s.t. $\forall q \in U$, $\varphi_\varepsilon := \varphi(-, q) : (-\varepsilon, \varepsilon) \to M$ is an integral*

*curve starting at $q$.*

*we call such $\varphi$ a local **flow** generated by $X$.*

**Definition 2.21.** Given $X \in \mathfrak{X}M$, a global **flow** generated by $X$ is a smooth map $\varphi : \mathbb{R} \times M \to M$ *s.t.* $\forall q \in M$, $\varphi_q := \varphi(-, q)$ is the maximal integral curve of $X$ starting at $q$.
$\Leftrightarrow \dfrac{\partial \varphi}{\partial t}(s, p) = X_{\varphi(s,p)}, \forall s \in \mathbb{R}, p \in M$ and $\varphi(0, p) = p, \forall p \in M$.

If such global flow exists, then we say $X$ is **complete**.

**Example 2.22.**

- $X = x \cdot \partial x \in \mathfrak{X}\mathbb{R}$ is complete, where global flow $\varphi : \mathbb{R} \times M \to M$, $\varphi(t, p) = p \cdot e^t$.
- $X = x^2 \partial x$ is not complete. Max integral curve starting at $1$ is given by $\gamma(t) = \dfrac{1}{1-t}, t \in (-\infty, 1) \neq \mathbb{R}$.

Given $X \in \mathfrak{X}M$, we define $\mathrm{Supp}X = \overline{\{p \in M : X_p \neq 0\}}$.

**Theorem 2.23.** *If a vector field $X$ is compactly supported, then $X$ is complete.*

**Corollary 2.24.** *Any vector field on closed manifold is complete.*

**Lemma 2.25** (Escaping lemma). *Suppose $\gamma : (a, b) \to M$ is a max integral curve, with $(a, b) \neq \mathbb{R}$. Then $\nexists$ compact $K \subset M$ s.t. $\gamma(a, b) \subset K$*

*Proof.* Otherwise, suppose $\gamma(a, b) \subset K$. WLOG, we may assume $b < +\infty$.

Take $(t_i) \to b$ from left. Then $\gamma(t_i) \in K$. After passing to subsequence, we may assume $(\gamma(t_i)) \to p \in K$.

Then $\exists$ $U$ Nbh of $p$, local flow $\varphi : (-\varepsilon, \varepsilon) \times U \to M$. Take $n$ large enough *s.t.* $b - t_n < \varepsilon, \gamma(t_n) \in U$. Then $\gamma(- + t_n) : (a - t_n, b - t_n) \to M$, $\varphi(-, \gamma(t_n)) : (-\varepsilon, \varepsilon) \to M$ are both integral curves for $X$ starting at $\gamma(t_n)$. By uniqueness, they coincide.

Let $\widehat{\gamma} : (a, t_n + \varepsilon) \to M$ be defined by $\widehat{\gamma}(t) = \begin{cases} \gamma(t), t \in (a,b) \\ \varphi(t - t_n, \gamma(t_n)), t \in [b, t_n + \varepsilon) \end{cases}$

Then $\widehat{\gamma}$ is an integral curve with larger domain, then $\gamma$ contradiction with the maxity of $\gamma$. □

*Proof of 2.23.* Take any max integral curve $\gamma : (a,b) \to M$. Suppose $(a,b) \neq \mathbb{R}$. Then $X_{\gamma(s)} \neq 0$, $\forall s$. Otherwise, the constant map $\mathbb{R} \to M, t \mapsto \gamma(s)$ is an integral curve with lager domain.

So $\forall s$, $\gamma(s) \in \mathrm{Supp}X \Rightarrow \gamma(a,b) \subset \mathrm{Supp}X$ which is compact $\Rightarrow (a,b) = \mathbb{R}$ by the lemma. This causes contradiction! □

A smooth $\varphi : \mathbb{R} \times M \to M$ is called an **one-parameter transformation group** if

(1) $\varphi_0 := \varphi(0, -) = \mathrm{id}_M$

(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}$. In particular, $\varphi_s^{-1} = \varphi_{-s}$.

**Theorem 2.26.** *$\varphi \in C^\infty(\mathbb{R} \times M, M)$, then $\varphi$ is an one-parameter transformation group if and only if $\varphi$ is the global flow generated by some $X \in \mathfrak{X}M$*

**Lemma 2.27** (Translation lemma)**.** *If $\gamma : (a,b) \to M$ is an integral curve for some $X \in \mathfrak{X}M$, then $\forall s \in \mathbb{R}$, $\gamma(- + s) : (a - s, b - s) \to M$ is also an integral curve for $X$.*

*Proof.* Let $\iota = \gamma(- + s)$. Then $\iota'(t) = X_{\gamma'(t+s)} = X_{\iota(t)}$ □

**Lemma 2.28.** *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_s \circ \varphi_r(p) = \varphi_{s+r}(p)$ provided that $s, t, s + t \in (-\varepsilon, \varepsilon), p, \varphi_r(p) \in U$.*

*Proof.* $\gamma_p = \varphi(-, p)$ is an integral curve for $X$.

$\Rightarrow \gamma_p(- + s)$ is an integral curve for $X$ starting at $\gamma_p(s) = \varphi_s(p)$. But $\gamma_{\varphi_s(p)}$ is also an integral curve starting at $\varphi_s(p)$. Thus $\gamma_{\varphi_s(p)} = \gamma_p(- + s) \Rightarrow \varphi_r \circ \varphi_s(p) = \gamma_{r+s}(p)$ □

**Lemma 2.29.** *Let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow for some $X \in \mathfrak{X}M$. Then $\varphi_{s,*}(X_p) = X_{\varphi_s(p)} \in T_{\varphi_s(p)}M$ i.e. any vector field is invariant under its flow.*

*Proof.* Take $f \in C^\infty_{\varphi(p)}(M)$.

$$\begin{aligned}\varphi_{s,*}(X_p)(f) &= X_p(f \circ \varphi_s)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_s(\varphi_t(p)))|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi_t(\varphi_s(p)))|_{t=0}\\ &= X_{\varphi_s(p)}(f)\end{aligned}$$

□

*Proof of 2.26.* "$\Leftarrow$" is because the lemma $\varphi_s \circ \varphi_r = \varphi_{s+r}$

"$\Rightarrow$" Let $X = \{X_p\}$ where $X_p = \frac{\partial \varphi}{\partial t}|_{(0,p)}$.

Leave it as an exercise. □

**Time dependent** vector field is a smooth map $X : \mathbb{R} \times M \to TM$ *s.t.* $X_{(t,p)} \in T_pM$.

A smooth curve $\gamma(a, b) \to M$ is the **integral curve** for $X$ if $\gamma'(t) = X_{(t,\gamma(t))}$.

In local chart, solving $\gamma$ is still solving ODE, so most results still holds for time dependent vector field. Those are some properties:

- Uniqueness: $\gamma_1, \gamma_2$ are both integral curves for $X$, $\gamma_1(c) = \gamma_2(c) \Rightarrow \gamma_1 \equiv \gamma_2$
- Max integral curve exists and is unique.
- Local flow exists.

Now define $\mathrm{Supp}X = \overline{\{p \in M : X_{t,p} \neq 0 \text{ for some } t\}}$.

Then $X$ is compactly supported, then $X$ is complete( *i.e.* a global flow $\varphi : \mathbb{R} \times M \to M$)

But something is not true for time dependent vector field:

- translation lemma is not true.
- vector field change under its flow.
- global flow can not implies one-parameter transformation group.

## 2.4 Another Definition of Vector Field

A derivation on $M$ is a $\mathbb{R}$-linear map $C^\infty(M) \xrightarrow{D} C^\infty(M)$ that satisfies the Leibniz rule:

$$D(f \cdot g) = Df \cdot g + f \cdot Dg$$

**Theorem 2.30.** *We have a bijection:*

$$\rho : \mathfrak{T}M \xrightarrow{1:1} \{\textit{derivation on } M\}$$

$$X \mapsto D_X : f \mapsto X(f)$$

**Lemma 2.31.** $D_p : \mathfrak{T}_p M \to$ *$\mathbb{R}$-linear map* $C^\infty(M) \to \mathbb{R}$ *s.t.* $D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g)$ *is an isomorphism of vector spaces.*

*Proof.* Leave it as an exercise. □

**Lemma 2.32.** *Given a vector field(not necessarily smooth)* $X = \{X_p\}_{p \in M}$ *, X is smooth* $\Leftrightarrow \forall f \in C^\infty(M), X(f)$ *is smooth.*

*Proof.* "$\Leftarrow$" $\forall p \in M$, take chart $(U, x^1, x^2, \cdots, x^n)$ around $p$. $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ $f^i : U \to \mathbb{R}$, where $f^i = X|_U(x^i)$. Take $\varphi : M \to [0,1]$ *s.t.* $\varphi \equiv 1$ near $p$, Supp$\varphi \subset U$,$\varphi \cdot x^i \in C^\infty(M)$.

Then $X(\varphi \cdot x^i) = f^i$ near $p$. By assumption, $f^i$ is smooth near $p$. So $f^i$ is smooth, so $X$ is smooth.

"$\Rightarrow$" Similar. $\square$

**Theorem 2.33.** *The map $\rho : \mathfrak{X}M \to \{\text{derivation on } M\}, X \mapsto (D_x : f \mapsto X(f))$ is well-defined and bijective.*

*Proof.* $\rho$ is well-defined: $X(f) \in C^\infty(M)$ by Lemma 2.32, and $D_x(fg) = D_x(f)g + fD_x(g)$ since $X$ is a point-derivation.

$\rho$ is injective: $D_x = D_y \Rightarrow D_{X_p} = D_{Y_p}$ as maps $C^\infty(M)$ to $\mathbb{R}$. By Lemma 2.31, we have $X_p = Y_p, \forall p$. So $X = Y$.

$\rho$ is surjective: Given $D : C^\infty(M) \to C^\infty(M)$. Define $D_p : C^\infty(M) \to \mathbb{R}$ by $D_p(f) := D(f)(p)$ satisfies the Leibniz rule. By Lemma 2.31, $D_p = D_{X_p}$ for some $X_p \in T_pM$. Define $X = \{X_p\}_{p\in M}$. Then $X(f) = D(f), \forall f \in C^\infty(M)$. By Lemma2.32, $X$ is a smooth vector field. $\square$

# 3 Lie group, Lie algebra and Lie bracket

## 3.1 Lie Bracket

In this section, we can actually find those identification:

$$\begin{aligned}\{\text{Tangent vector at } p\} &= \{\text{point derivation at } p\} \\ &= \{\mathbb{R}\text{-linear maps } C_p^\infty(M) \xrightarrow{D_p} \mathbb{R} \quad s.t. \\ &\quad D_p(fg) = D_p(f)g(p) + f(p)D_p(g)\}\end{aligned}$$

$$\begin{aligned}\{\text{smooth vector fields}\} &= \{\text{smooth sections of } TM\} \\ &= \{\text{derivation on } M\}\end{aligned}$$

**Notation 3.1.** We will identify $X \in \mathfrak{X}M$ with its derivation $D_x : C^\infty(M) \to C^\infty(M)$. So a vector field is just a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M) \quad s.t.\ X(fg) = fX(g) + X(f)g$.

**Definition 3.2** (Lie bracket)**.** Given two (smooth) vector field $X, Y : C^\infty(M) \to C^\infty(M)$, we define the **Lie bracket**

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

**Theorem 3.3.** *For any $X, Y \in \mathfrak{X}M$, $[X, Y] \in \mathfrak{X}M$*

*Proof.* Easy to check that $[X, Y]$ is linear.

By Leibuniz rule,

$$\begin{aligned} [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg) \\ &= (X \cdot Y)(f) \cdot g + f \cdot (X \circ Y)(g) - (Y \cdot X)(g) - f \cdot ((Y \circ X)(g)) \\ &= [X, Y](f) \cdot g - f \cdot [X, Y](g) \end{aligned}$$

□

So What is the geometric meaning of $[X, Y]$? Non commutatiy of flows.

**Fact 3.4.** *Given $X, Y \in \mathfrak{X}M$, we say $X, Y$ are commutative vector field if $[X, Y] = 0$*

*$X, Y$ are commuative iff for any local flows $\varphi^X : (-\varepsilon, \varepsilon) \times U \to M$, $\varphi^Y : (-\varepsilon, \varepsilon) \times U \to M$ we have $\varphi^X_s \circ \varphi^T_t = \varphi^Y_t \circ \varphi^X_s$*

**Proposition 3.5** (Calculation of $[V, W]$ using local charts). *Chart* $(U, x^1, \cdots, x^n)$, $V, W \in \mathfrak{X}M$, $V|_U = \sum_{i=1}^{n} V^i \frac{\partial}{\partial x^i}$, $W|_U = \sum_{i=1}^{n} W^i \frac{\partial}{\partial x^i}$. *Then*

$$
\begin{aligned}
[V, W]|_U &= \sum_{i=1}^{n} (V(W^i) - W(V^i)) \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{n} V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i} \\
&= \sum_{1 \leqslant i,j \leqslant n} (V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}) \frac{\partial}{\partial x^i}
\end{aligned}
$$

**Example 3.6.** $V = x\partial x + y\partial y$, $W = -y\partial x + x\partial y$ commutes.

**Proposition 3.7** (Properties of Lie bracket).

*(a)* *Natuality under push-forword.*

*Given any* $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, *we have* $[F_* V, F_* W] = F_*[V, W]$.

*(b)* $\mathbb{R}$*-linearity* $\forall a, b \in \mathbb{R}$

$$
[aX + bV, W] = a[X, W] + b[V, W]
$$

$$
[W, aX + bV] = b[W, X] + a[W, V]
$$

*(c)* *anti-symmetric* $[V, W] = -[W, V]$

*(d)* *Jacobi identity*

$$
[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0
$$

(f) *Leibuniz rule*

$$[fV, gW] = fg[V, W] + (f \cdot Vg)W - (g \cdot Wf)V$$

**Definition 3.8.** Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M$, $W \in \mathfrak{X}N$. We say $W$ is $F$**-related** to $V$ if $\forall p \in M$, $F_{p,*}(V_p) = W_{F(p)}$ where $F_{p,*} : T_pM \to T_{f(p)}N$

**Example 3.9.** $F : S^1 \to \mathbb{R}^2, \theta \mapsto (\cos\theta, \sin\theta)$, $V = \partial\theta, W = -y\partial x + x\partial y$.

*Note* 1. In general, given $V \in \mathfrak{X}M$ and $F \in C^\infty(M, N)$. There may not exist $W \in \mathfrak{X}M$ *s.t.* $V, W$ are $F$-related. Even such $W$ exists, it may not be unique.

However, if $F$ is a diffeomorphism, given any $V$, $\exists$ unique $W$ *s.t.* $V$ and $W$ are $F$-related. Actually, $W_p = F_* V_{F^{-1}(p)}$.

Such $W$ is called **push forward** of $V$ along $F$, denoted by $F_*V$, only defined when $F$ is a diffeomorphism.

**Lemma 3.10.** *$\forall V \in \mathfrak{X}M, W \in \mathfrak{X}N$, $F \in C^\infty(M, N)$. Then $W$ is $F$-related to $V$ iff $\forall f \in C^\infty(N)$, $V(f \circ F) = W(f) \circ F \in C^\infty(M)$*

*Proof.* Check that $F_{p,*}(V_p)(f) = W_{F(p)}(f), \forall f \in C^\infty(N)$ □

**Proposition 3.11.** *Given $V_0, V_1 \in \mathfrak{X}M$, $W_0, W_1 \in \mathfrak{X}N$, $F \in C^\infty(M, N)$, $W_i$ is $F$-related to $V_i$, $i = 0, 1 \Rightarrow [W_0, W_1]$ is $F$-related to $[V_0, V_1]$*

**Corollary 3.12** (Naturality of Lie bracket)**.** *Given any $F \in \mathrm{Diff}(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}M$, we have $[F_*V, F_*W] = F_*[V, W]$*

The rest of Proposition 3.7 is easy to check if it is viewed as a mapping $C^\infty(M) \to C^\infty(M)$.

## 3.2 Lie Algebra of a Lie Group

**Definition 3.13.** A **Lie algebra** $g$ is $\mathbb{R}$-linear space $g$ with map $[-,-] : g \times g \to g$ *s.t.* it is bilinear, anti-symmetric and satisfies the Jacobian identity.

Then $(\mathfrak{X}M, [-,-])$ is an infinite dimensional Lie algebra.

For $G$ Lie group, $\forall g \in G$ we have diffeomorphism

$$l^g : G \to G, h \mapsto gh$$

$$r^g : G \to G, h \mapsto hg$$

We say $X \in \mathfrak{X}G$ is **left invariant** if $l^g_*(X) = X$, $\forall g \in G$. Similarly, $X$ is **right invariant** if $r^g_*(X) = X$.

**Proposition 3.14.** *$X, Y$ are left/right invariant $\Rightarrow$ $[X, Y]$ is left/right invariant.*

*Proof.* $l^g_*[X, Y] = [l^g_* X, l^g_* Y] = [X, Y]$ □

So we can find a natural Lie algebra of $G$:

$$\mathrm{Lie}(G) := \{\text{left invariant vector fields on } G\}, \text{with } [-,-] \text{ restricted from } \mathfrak{X}G$$

**Theorem 3.15.** *Given any $V \in T_e G$, $\exists$ unique left invariant $\widehat{V} \in \mathfrak{X}G$ s.t. $\widehat{V}_e = V$.*

**Corollary 3.16.** $\mathrm{Lie}(G) \cong T_e G$ *as vector spaces.*

*Proof of Theorem 3.15.*

**Uniqueness of $\widehat{V}$:** $\widehat{V}_g = l^g_{e,*}(\widehat{V}_e) = l^g_{e,*}(V)$. So $\widehat{V}$ is determined by $V$.

**Existence of $\widehat{V}$:** Let $\widehat{V} = \{\widehat{V}_g\}_{g \in G}$ where $\widehat{V}_g = l^g_{e,*}(\widehat{V}_e)$.

$\widehat{V}$ is left-invariant because

$$(l^h_*(\widehat{V}))_g = l^h_{h^{-1}g,*}(\widehat{V}_{h^{-1}g}) = l^h_{h^{-1}g,*}(l^{h^{-1}g}_{e,*}(V)) = l^g_{e,*}(V) = \widehat{V}_g$$

$\widehat{V}$ is smooth: Take any $f \in C^\infty(G)$ suffices to show $\widehat{V}(f) \in C^\infty(G)$.

Take any smooth $\gamma : \mathbb{R} \to G \quad s.t. \ \gamma(0) = e, \gamma'(0) = V$. Then $l^g \circ \gamma : \mathbb{R} \to G$ satisfies $l^g \circ \gamma(0) = g, (l^g \circ \gamma)(0) = g, (l^g \circ \gamma)'(0) = l^g_{e,*}(V) = \widehat{V}_g$

So

$$\widehat{V}(f)(g) = \widehat{V}_g(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(l^g \circ \gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(g \cdot \gamma(t))|_{t=0} \tag{3.1}$$

Consider the map

$$\widehat{f} : G \times \mathbb{R} \xrightarrow{\mathrm{id}\times\gamma} G \times G \xrightarrow{\cdot} G \xrightarrow{f} \mathbb{R}$$

$$(g,t) \mapsto (g, \gamma(t)) \mapsto g \cdot \gamma(t) \mapsto f(g \cdot \gamma(t))$$

Then $\widehat{f}$ is smooth, $\frac{\partial \widehat{f}}{\partial t}|_{t=0} : G \to \mathbb{R}$ is smooth, but $\frac{\partial f}{\partial t}|_{t=0}(g) = \widehat{V}(f)(g)$ by 3.1. So $\widehat{V}(f) \in C^\infty(G)$. □

**Example 3.17.** $G = \mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det A \neq 0\} \subset M_n(\mathbb{R}) \cong \mathbb{R}^2$.

$\mathrm{gl}(n, \mathbb{R})$= $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = T_I\mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$

**Theorem 3.18.** $\forall A, B \in \mathrm{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$, $[A, B] = AB - BA$.

**Remark 3.19.** This theorem shows that the Lie bracket viewed as the Lie algebra and matrix are the same. In some sense, it means the Lie bracket defined in three sets $\mathrm{gl}(n, \mathbb{R}) = T_I\mathrm{GL}(n, \mathbb{R}) = M_n(\mathbb{R})$ can commute with those corresponding, or equivalently, are just the same.

**Lemma 3.20.** $\forall A \in \mathrm{gl}(n, \mathbb{R})$, *the left invariant vector field* $\widehat{A}$ *is complete and generate the flow* $\varphi_t : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}), \varphi_t(g) = ge^{At} = g(I + At + \frac{A^2t^2}{2!} + \cdots)$

*Proof.*

$$\widehat{A}_g = g \cdot A \in T_g G = M_n(\mathbb{R})$$

$$\frac{\partial}{\partial t}\varphi_t(g) = \frac{\partial}{\partial t}(g(e^{At})) = ge^{At}A = \widehat{A}_{g\cdot e^{At}} = \widehat{A}_{\varphi_t(g)}$$

□

**Remark 3.21.** This lemma tells how to compute $A(f) = \widehat{A}(f)(I)$ as a tangent vector or a vector field, as we will see in the next proof.

*Proof of Theorem 3.18.* Take $A, B \in \mathrm{gl}(n,\mathbb{R})$. Want to show $[\widehat{A}, \widehat{B}]_I = AB - BA$.

Pick $f \in C_I^\infty(G)$, need to show $A(\widehat{B}(f)) - B(\widehat{A}(f)) = (AB - BA)(f)$

Further Simplification: Just need to focus on $f = x^{ij}$, where $x^{ij} : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}, E \mapsto (E - I)_{ij}$. Actually, $\partial x^{ij}$ is what we choose as a basis of $T_I\mathrm{GL}(n,\mathbb{R})$.

Such $f$ satisfies $f(I + -)$ is $\mathbb{R}$-linear.

Recall that Given $W \in \mathfrak{X}M$, $W(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t}f(\varphi_t^W(p))|_{t=0}$.

So $\widehat{B}(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}f(ge^{Bt})|_{t=0}$.

So since $A(\widehat{B}(f)) = \widehat{A}((\widehat{B})(f))(I) = \frac{\mathrm{d}}{\mathrm{d}s}(\widehat{B}(f)(e^{As}))|_{s=0}$,

$$A(\widehat{B}(f)) = \frac{\mathrm{d}}{\mathrm{d}s}(\widehat{B}(f)(e^{As}))|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}f(I+sA+tB+\frac{s^2}{2}A^2+stAB+\frac{t^2}{2}B^2+\cdots)|_{s=t=0}$$

Similarly,

$$B(\widehat{A}(f)) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}f(I + sA + tB + \frac{s^2}{2}A^2 + stBA + \frac{t^2}{2}B^2 + \cdots)|_{s=t=0}$$

So $A(\widehat{B}(f)) - B(\widehat{A}(f)) = f(I + (AB - BA)) = (AB - BA)(f)$ since $f$ is $\mathbb{R}$-linear. □

Similarly, for $G = \mathrm{GL}(n,\mathbb{C}), \mathrm{Lie}(G) = \mathrm{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$, we have $[A, B] = AB - BA$.

Actually, we have those properties of Lie group and Lie algebra.

- Any simply connected Lie group are determined by its Lie algebra.
- Given any connected Lie group $G$, its universal cover $\widehat{G}$ is simply-connected with $\pi^{-1}(G) \subset Z(\widehat{G})$.

What is the meaning of Lie bracket. There is a fact about it:

**Fact 3.22.** *$G$ is connected Lie group. $G$ is abelian iff* $[-,-] = 0$ *on* $\mathrm{Lie}(G)$

## 3.3 Morphisms between Lie group and Lie algebras

A smooth map $F : G \to H$ between two Lie group is called a **morphism** if $F(gh) = F(g)F(h)$.

A linear map $L : g \to h$ between Lie algebra is called a **morphism** if $L[u,v] = [Lu, Lv]$.

**Proposition 3.23.** *Let $F : G \to H$ be a morphism of Lie groups. Then $F_{e,*} : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a morphism of Lie algebra.*

*Proof.* $V_0, V_1 \in \mathrm{Lie}(G) = T_eG$, $W_i = F_{e,*}(V_i) \in \mathrm{Lie}(H) = T_eH$. Let $\widehat{V}, \widehat{W}$ be left-invariant vector fields.

*Claim.* $\widehat{W_i}$ is $F$-compatible with $\widehat{V_i}$ for $i = 0, 1$.

*Proof of Claim.* $\forall g \in G$, $F_*(\widehat{V}_g) = F_*(l^g_*(V)) = (F \circ l^g)_*(V) = (l^{F(g)} \circ F)_*(V) = l^{F(g)}(W) = \widehat{W}_{F(g)}$ □

So $[\widehat{W_0}, \widehat{W_1}]$ is $F$-compatible with $[\widehat{V_0}, \widehat{V_1}]$. In particular, $[W_0, W_1] = F_*([V_0, V_1])$. □

# 4 Vector Field

## 4.1 Canonical form of a Field

Recall that $V \in \mathfrak{X}M$, $p \in M$ is called a **regular point** if $V_p \neq 0$, and is called a **singular point** if $V_p = 0$.

**Theorem 4.1** (Canonical Form Theorem). *Let $p$ be a regular point of $V$. Then $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ s.t. $V|_U = \partial x^1$*

*Proof.* This is a local problem. We may assume $M \subset \mathbb{R}^n$ open. We may also assume $p = 0, V_0 = \partial r^1|_0$ where $r^i$ coordinate function.

Let $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)^n \to M$ be the local flow of $V$.

Define $\psi : (-\varepsilon, \varepsilon)^n \to M$ by $\psi(t, r^2, \cdots, r^n) = \varphi(t, (0, r^2, \cdots, r^n))$. Then $\psi(-, r^2, \cdots, r^n)$ is an integral curve for $V$. Therefore, $\psi_*(\partial t) = V$.

At $\vec{0}$, we have $\psi_{\vec{0},*}(\partial t) = V_{\vec{0}} = \partial r^1$, $\psi_{\vec{0},*}(\partial r^i) = \partial r^i$.

So $\psi_{*,\vec{0}} : T_{\vec{0}}(-\varepsilon, \varepsilon)^n \to T_{\vec{0}}M$ is an isomorphism.

By the inverse function theorem, $\exists U' \subset (-\varepsilon, \varepsilon)^n$, $U \subset M$ s.t. $\psi|_{U'} : U' \to U$ is a diffeomorphism.

Then $(U, (\psi|_{U'})^{-1})$ is the local chart what we need. □

**Remark 4.2.** Regular point in a vector field is simple, as we can view it in the standard chart locally. However, behavior of $V$ art a singular point can be complicated. For example, for $f(x, y) = x^2 - y^2$, $\nabla f = 2x\partial x - 2y\partial y$, $g : \mathbb{C} \to C, z \mapsto z^n$, they behave differently at $\vec{0}$.

## 4.2 Lie Derivative of Vector Field

$V, W \in \mathfrak{X}M$, $\mathcal{L}_V W$ is the directional derivative of $W$ in the direction of $V$.

**Definition 4.3.** The **Lie derivative** $\mathcal{L}_V W \in \mathfrak{X}M$ is defined as follows: $\forall p \in M$, let $\{\theta_t : U \to M\}_{t\in(-\varepsilon,\varepsilon)}$ be the local flow for $V$. Then

$$(\mathcal{L}_V W)_p = \lim_{t\to 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}$$

**Remark 4.4.** This defintion is actually a difference between $T_{\theta_t(p)}$ and $T_p$, which need pullback.

**Lemma 4.5.** *$\mathcal{L}_V W$ is well-defined and smooth.*

*Proof.* For $p \in M$, take local chart $(U, x^1, \cdots, x^n)$. Let $\theta : (-\varepsilon, \varepsilon) \times U \to M$ be the flow of $V$. Take $J_0 \subset (-\varepsilon,\varepsilon)$, $U_0 \subset U$. Let $\theta^i = x^i \circ \theta : J_0 \times U_0 \to \mathbb{R}$, $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Under the basis $\{\partial x^i\}$, $(\theta_{-t})_* : T_{\theta_t(p)}M \to T_pM$ is represented by

$$\left(\frac{\partial\theta^i(-t,\theta(t,x))}{\partial x^j}\right)_{i,j}$$

So $(\theta_{-t})_* W_{\theta_t(x)} = \sum_{i,j} \frac{\partial\theta^i(-t,\theta(t,x))}{\partial x^j} W^j(\theta(t,x)) \cdot \partial x^i$ is smooth in $t, x$. So

$$(\mathcal{L}_V W)_x = \frac{\partial((\theta_{-t})_*(W_{\theta_t(x)}))}{\partial t}|_{t=0}$$

is well-defined and smooth. □

**Theorem 4.6.** *For all $V, W \in \mathfrak{X}M$, $\mathcal{L}_V W = [V, W]$.*

*Proof.* For $p$ is a regular point of $V$. By canonical form theorem 4.1, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ *s.t.* $V|_U = \partial x^1$. Let $W|_U = \sum_{i=1}^n W^i \partial x^i$.

Then $\theta_t(x_1, \cdots, x_n) = (x_1 + t, x_2, \cdots, x_n)$. So

$$\mathcal{L}_V W|_U = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

.

$$[V,W]|_U = \sum_i V(W^i)\partial x^i - \sum_i W(V^i)\partial x^i = \sum_i \frac{\partial W^i}{\partial x^1} \cdot \partial x^i$$

Then $[V,W]|_U = \mathcal{L}_V W$.

For $p$ is a singular point but $p \in \operatorname{Supp}(V)$. Then $\exists\ p_i \to p \quad s.t.\ V_p \neq 0$. By the previous case $(\mathcal{L}_V W)_{p_i} = [V,W]|_{p_i}$. By continuity, We have $(\mathcal{L}_V W)_p = [V,W]_p$.

For $p \notin \operatorname{Supp}(V)$, $\exists$ Nbd $U$ of $p \quad s.t.\ V|_U = 0$. Then $\theta_t(q) = q$. So

$$(\mathcal{L}_V W)|_U = 0 = [V,W]|_U$$

□

**Corollary 4.7.**

- *$\mathcal{L}_V W$ is $\mathbb{R}$-linear with respect to $V, W$.*
- *$\mathcal{L}_V W = -\mathcal{L}_W V$.*
- *(Jacobian identity) $\mathcal{L}_V[W,X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.*
- *(Jacobian identity) $\mathcal{L}_{[V,W]}X = \mathcal{L}_V\mathcal{L}_W X - \mathcal{L}_W\mathcal{L}_V X$.*
- *$\mathcal{L}_V(fW) = (Vf)\cdot W + f\mathcal{L}_V W$*
- *Let $F : M \to N$ be a diffeomorphism. Then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_*(V)} F_*(W)$.*

## 4.3 Commuting Vector Fields

**Definition 4.8.** We say $V, W \in \mathfrak{X}M$ **commutes** if $[V,W] = 0$.

**Theorem 4.9.** *TFAE:*

*1 $V, W$ commutes.*

2 *$W$ is invariant under the flow generated by $V$, i.e. $\theta_{t,*}(W_p) = W_{\theta_t(p)}$*

3 *The flow for $V, W$ commutes, i.e. $\theta_t \circ \eta_s = \eta_s \circ \theta_t$ whenever either side is defined or equivalently, whose the domain is compatible.*

**Lemma 4.10.** *Given $F \in C^\infty(M, N)$, $V \in \mathfrak{X}M, W \in \mathfrak{X}N$. Then $W$ is $F$-related to $V$ if and only if $\forall t \in \mathbb{R}$, $\eta_t \circ F = F \circ \theta_t$ on the domain of $\theta_t$, which means*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow{\scriptstyle \theta_t} & & \downarrow{\scriptstyle \eta_t} \\ M & \xrightarrow{F} & N \end{array} \text{ commutes.}$$

*Proof.* "$\Rightarrow$" Let $\gamma = F \circ \theta^p : J \to N$ satisfies

$$\gamma'(t) = (F \circ \theta^p)'(t) = F_*((\theta^p)'(t)) = F_*(V_{\theta^p(t)}) = W_{F(\theta^p(t))} = W_{\gamma(t)}$$

So $\gamma$ is an inetgral curve of $W$ starting at $\gamma(0) = F(p)$ *i.e.* $F \circ \theta^p = \gamma(t) = \eta^{F(p)}(t)$ *i.e.* $F \circ \theta_t = \eta \circ F$.

"$\Leftarrow$" Suppose $F \circ \theta_t = \eta \circ F$. Then $(F \circ \theta^p)(t) = \eta^{F(p)}(t)$.

Then $F_* V_p = F_*((\theta^p)'(0)) = (F \circ \theta^p)'(0) = (\eta^{F(p)})'(0) = W_{F(p)}$. So $W$ is $F$-related to $V$. □

*Proof of Theorem 4.9.* 2 $\Rightarrow$ 1: $(\theta_{-t})_*(W_{\theta_t(p)}) = W_p$. So

$$\mathcal{L}_V W = \lim_{t \to 0} \frac{(\theta_{-t})_*(W_{\theta_t(p)}) - W_p}{t} = 0$$

1 $\Rightarrow$ 2: Let $X(t) = (\theta_{-t})_*(W_{\theta_t(p)})$, $p \in M$.

Want to show that $X(t) = X_p$ for all $t$. Suffices to show $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) = 0$.

For $t_0 = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} X(t) = (\mathcal{L}_V W)_p = 0$.

In general, set $s = t - t_0$, $X(t) = (\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} X(t) = \frac{\mathrm{d}}{\mathrm{d}s}|_s X(s + t_0)$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}s}|_s(\theta_{-t_0})_* \circ (\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_* \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(\theta_{-s})_*(W_{\theta_s(\theta_{t_0}(p))}) \\
&= (\theta_{t_0})_*(\mathcal{L}_V W)_{\theta_{t_0}(p)} \\
&= 0
\end{aligned}
$$

$2 \Rightarrow 3$. For simplicity, assume $V, W$ are complete. $F = \theta_s : M \to M$. By 2, $W$ is $F$-related to $W$. So by the lemma,

$$
\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\downarrow{\scriptstyle \theta_t} & & \downarrow{\scriptstyle \eta_t} \\
M & \xrightarrow{F} & M
\end{array}
\quad \text{commutes.}
$$

$\eta_t$ is flow for $W$. *i.e.* $\theta_s \circ \eta_t = \eta \circ \theta_s$

$3 \Rightarrow 2$ is similar. The diagram commutes, so $W$ is $F$-related to $W$. □

### 4.3.1 Canonical Form of Commuting Vector Field

**Theorem 4.11.** *Given $V_1, \cdots, V_k \in \mathfrak{X}M$, s.t.*

*1)* $[V_i, V_j] = 0, \forall i, j.$

*2)* $V_{1,p}, V_{2,p}, \cdots, V_{k,p}$ *linearly independent at some* $p \in M$

*Then* $\exists$ *local chart* $(U, x^1, \cdots, x^n)$ *around* $p$ *s.t.* $V_i|_U = \dfrac{\partial}{\partial x^i}, \forall 1 \leqslant i \leqslant k$

We prove it using the inverse function theorem.

*Proof.* This is a local problem. So we may assume $M \subset \mathbb{R}^m$ be open with coordinate function $r^i : M \to \mathbb{R}, 1 \leqslant i \leqslant m$.

After translation and linear transformation, we may assume $p = \vec{0}$, $V_{i,\vec{0}} = \left.\dfrac{\partial}{\partial x^i}\right|_{\vec{0}}, 1 \leqslant i \leqslant k.$

Take local flow $\{\theta_t^i : (-\varepsilon, \varepsilon)^m \to M\}_{t \in (-\varepsilon,\varepsilon)}$ for $V_i$.

Define $\psi : (-\varepsilon, \varepsilon)^k \times (-\varepsilon, \varepsilon)^{m-k} \to M$, $\psi(t^1, \cdots, t^k, r^{k+1}, \cdots, r^m) = \theta^1_{t_1} \circ \theta^2_{t_2} \cdots \circ \theta^k_{t_k}(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$, where $\theta^i$ commutes with each other.

So if we fix $t^j, j \neq i$ except $t^i$, $\psi(t^1, \cdots, t^{i-1}, -, t^{i+1}, \cdots, t^k, r^{k+1}, \cdots, r^m)$ is an integral curve for $V^i$. Then $V^i$ is $\psi$-related to $\partial t^i$.

On the other hand. $\psi(0, 0, \cdots, 0, r^{k+1}, \cdots, r^m) = (0, 0, \cdots, 0, r^{k+1}, \cdots, r^m)$. So $\psi_{\vec{0},*} : T_{\vec{0},*} : T_{\vec{0}}(-\varepsilon', \varepsilon')^m \to T_{\vec{0}}M, \partial t^i \mapsto V_{i,0} = \partial x^i|_0$ and $\partial r^i \mapsto \partial r^i, k+1 \leqslant i \leqslant m$.

So $\psi_{\vec{0},*}$ is an isomorphism.

By the inverse function theorem, there exists Nbh $U' \subset (-\varepsilon', \varepsilon')^m$ *s.t.* $\psi : U' \to U$ is a diffeomorphism and $U \subset M$ open.

Then $(U, (\psi|_U)^{-1})$ is the local chart we need. □

## 4.4 The Constant Rank Theorem

$F \in C^\infty(M, N)$, $p \in M$. The **rank** of $F$ at $p$ is

$$\begin{aligned}\mathrm{rank}_p F &:= \mathrm{rank}(F_{p,*} : T_pM \to T_{F(p)}N) \\ &= \mathrm{rank}\left(\frac{\partial F^i(p)}{\partial x^j}\right)_{i,j}\end{aligned}$$

We say $F$ has **constant rank** $k$ near $p$ if $\exists$ Nbh $U$ of $p$ *s.t.* $\mathrm{rank}_q F = k, \forall q \in U$

**Proposition 4.12.**

$$\mathrm{rank}_q(F) \leqslant \min(\dim(M), \dim(N))$$

**Theorem 4.13** (The constant rank theorem)**.** *Suppose $F : M \to N$ has constant rank $k$ near $p \in M$, then $\exists$ local charts $U \xrightarrow[\cong]{\varphi} \mathbb{R}^m$ around $p$, $V \xrightarrow[\cong]{\psi} \mathbb{R}^n$ around $F(p)$ s.t.*

$$\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \textit{ is given by } (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0)$$

*Proof.* This is a local problem. So we may assume $M = \mathbb{R}^m, N = \mathbb{R}^n$ by restricting to local charts. And $p = 0, F(p) = 0$. After changing orders of coordinates, may assume $\left(\frac{\partial F^i}{\partial x^j}(0)\right)_{1\leqslant i,j\leqslant k}$ is invertible. Write $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}, \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Then $F(x,y) = (Q(x,y), R(x,y))$. Consider $\varphi : \mathbb{R}^m \to \mathbb{R}^m, (x,y) \mapsto (Q(x,y), y)$. Then

$$\varphi_{(0,0),*} = \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0) & 0 \\ & \\ \frac{\partial Q^i}{\partial y^j}(0) & I_{m-k} \end{bmatrix} \tag{4.1}$$

is invertible.

By inverse function theorem, $\exists$ Nbh $U_0 \subset \mathbb{R}^m$, $\widetilde{U}_0 \subset \mathbb{R}^m$ of $0$ $\ s.t.\ \varphi : U_0 \to \widetilde{U}_0$ is a diffeomorphism.

$$\widetilde{U}_0 \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} U_0 \xrightarrow{F} \mathbb{R}^n$$

$$(Q(x,y), y) \leftarrow\!\shortmid (x,y) \mapsto (Q(x,y), R(x,y))$$

So $F \circ \varphi^{-1} : \widetilde{U}_i \to \mathbb{R}^n, (x,y) \mapsto (x, A(x,y))$. And

$$(F \circ \varphi^{-1})_{p,*} = \begin{bmatrix} I_k & 0 \\ & \\ \frac{\partial A}{\partial x}(p) & \frac{\partial A}{\partial y}(p) \end{bmatrix} \tag{4.2}$$

Since $\operatorname{rank}(F \circ \varphi^{-1})$ is k, $\frac{\partial A}{\partial y}(p) = 0$. $\ i.e.\ A(x,y) = A(x)$.

We can find a map $\psi : (x,y) \mapsto (x, y - A(x))$ in a smaller neighborhood of $0$ by the inverse theorem similarly.

And $\psi \circ F \circ \varphi$ maps $(x,y)$ to $(x,0)$. So we end the proof. $\square$

**Definition 4.14.** $F \in C^\infty(M,N)$.

We say $F$ is **submersion** if $F_{p,*}$ is surjective $\forall p \in M$.

We say $F$ is **immersion** if $F_{p,*}$ is injective $\forall p \in M$.

We say $F$ is **embedding** if $F$ is immersion and $F$ is a topological embedding.(i.e. $F : M \to F(M)$ is a homeomorphism)

If $F$ is embedding(immersion resp.), we say $M$ or $F(M)$ is an **embedded submanifold**(**immersed submanifold**, resp.) of $N$.

Denote $M \looparrowright N$ be the immersion. $M \hookrightarrow N$ be the embedding.

**Example 4.15.**

- There is an example $F : S^1 \to \mathbb{R}^2$ where $F$ is an immersion but not an embedding.
- Projection $M \times N \to M$ is a submersion.
- $E \xrightarrow{p} B$ is a smooth vector bundle, then $p$ is a submersion.
- $\gamma : \mathbb{R} \to M$ is an immersion $\Leftrightarrow \gamma'(t) \neq 0$, $\forall t$.
- There is an example $\gamma : \mathbb{R} \to \mathbb{R}^2$ is injective immersion but not an embedding
- $\gamma : \mathbb{R} \to \mathrm{T}^2 = \mathbb{R}/_{\mathbb{Z}} \times \mathbb{R}/_{\mathbb{Z}}$, $x \mapsto (x, cx)$, $c \notin \mathbb{Q}$ is injective immersion but not embedding.

**Definition 4.16.** For $F : X \to Y$, we say $F$ is **proper** if for any compact set $K \subset N$, $F^{-1}(K)$ is compact.

**Lemma 4.17.** *$X$ is compact, $Y$ Hausdorff, then $F : X \to Y$ is proper.*

**Proposition 4.18.** *$F \in C^\infty(M, N)$ is an injective immersion, and $F$ is proper. Then $F$ is an embedding.*

*Proof.* $F : M \to F(M)$ is a closed map. □

**Definition 4.19.** For $F \in C^\infty(M, N)$.

$p \in M$ is called **regular point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is surjective.

$p \in M$ is called **critical point** if $F_{p,*} : T_pM \to T_{F(p)}N$ is not surjective.

$q \in N$ is called **regular value** if $\forall p \in F^{-1}(q)$, $p$ is a regular point.

$q \in N$ is called **critical value**(or **singular value**) if $\exists p \in F^{-1}(q)$, $p$ is a critical point.

**Theorem 4.20** (Sard). *Singular value has measure* $0$*.*

*Proof.* We will not prove it in this lecture. □

**Theorem 4.21.** *$M$ is an embedded submanifold of $N$ if and only if $\forall p \in M \subset N$, $\exists$ local chart $(U, x^1, \cdots, x^n)$ around $p$ of $N$ s.t. $M \cap U = \{(x^1, \cdots, x^m, 0, \cdots, 0)\}$*

*Proof.* "$\Rightarrow$": $F : M \to N$ is embedding $\Rightarrow$ $F$ has constant rank $m$. Apply constant rank theorem near $p$, and we finish the proof of "$\Rightarrow$"

The converse is trivial.

□

**Theorem 4.22.** *$F \in C^\infty(M, N)$, $q$ is a regular value of $F$. Then $F^{-1}(q)$ is an embedded submanifold of $M$. And*

$$\forall p \in F^{-1}(q), T_pF^{-1}(q) = \ker(F_{p,*} : T_pM \to T_{F(p)}N)$$

*Proof.* $q$ is regular value $\Rightarrow$ $\mathrm{rank}_p F = n$, $\forall p \in F^{-1}(q)$.

$\Rightarrow$ $\mathrm{rank}_{p'} F = n$, $\forall p'$ near $p$, since we know the rank of $p'$ near $p$ should not be less than that of $p$

So by the constant rank theorem, $F^{-1}(q)$ is a submanifold near $p$.

□

Denote

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \{A \in M_n(\mathbb{R}) | A + A^T = 0\}$$

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) | A + A^* = 0\}$$

$$\mathfrak{su}(n) = \{A \in u(n) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \mathrm{tr} A = 0\}$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) | \mathrm{tr} A = 0\}$$

**Theorem 4.23.** *Those above sets are the Lie algebra of the corresponding Lie group. For instance,* $\mathfrak{su}(n) = \mathrm{Lie}(\mathrm{SU}(n))$.

# 5 Differential forms

## 5.1 Introduction

Our goal is to define the integration $\displaystyle\int_M \alpha$ *s.t.*

- Works for any smooth manifold $M$, without embedding $M$ into $\mathbb{R}^n$
- Generalize two types of surface integral, *i.e.* $\int_\Sigma f \mathrm{d}S$ and $\int_\Sigma f \mathrm{d}x \wedge \mathrm{d}y$

For Cartan's idea, $\alpha$ is a "differential $k$-form" on $M$ *s.t.*

- $\forall F \in C^\infty(N, M)$, $F^*(\alpha)$ is a $k$-form on $N$
- If $k = \dim M$, then $\int_M \alpha \in \mathbb{R}$

## 5.2 Alternating Vector Linear Algebra

For $V_1, \cdots, V_n, W$ be $\mathbb{R}$-vector spaces, $f : V_1 \times \cdots \times V_n \to W$ is called **multi $\mathbb{R}$-linear** if

$$\begin{aligned} f(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_n) &= af(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_n) \\ &\quad + bf(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_n) \end{aligned} \tag{5.1}$$

**Example 5.1.**

- Inner product $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\cdot} \mathbb{R}$.
- Matrix multiplication $M_{n\times m}(\mathbb{R}) \times M_{m\times k}(\mathbb{R}) \to M_{n\times k}(\mathbb{R})$.
- Cross product $\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\times} \mathbb{R}^3$.
- Bilinear form.

We hope that we can construct a vector space $V_1 \otimes \cdots \otimes V_n \quad s.t.$ we have canonical isomorphism:

$$\{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\text{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \tag{5.2}$$

Then we can transform the study of multilinear algebra into the study of the normal linear algebra.

For any set $S$, let

$$\mathbb{R}\langle S\rangle = \left\{\text{formal linear combination } \sum_{i=1}^{n} a_i s_i | a_i \in \mathbb{R}, s_i \in S, n < \infty\right\} \tag{5.3}$$

Consider $\mathbb{R}\langle V_1 \times \cdots \times V_n \rangle = \left\{ \sum_{i=1}^{k} a^i (V_{i,1}, \cdots, V_{i,n}) | a^i \in \mathbb{R}, v_{i,j} \in V_j \right\}$. Denote

$$W = \mathrm{Span}\{(\cdots, av_i + bv_i', \cdots) - a(\cdots, v_i, \cdots) - b(\cdots, v_i', \cdots) | a, b \in \mathbb{R}, v_i, v_i' \in V_i\} \tag{5.4}$$

Define $V_1 \otimes \cdots \otimes V_n = \mathbb{R}\langle V_1 \times \cdots \times V_n \rangle /_W$, write $[(v_1, \cdots, v_n)]$ as $v_1 \otimes \cdots \otimes v_n$, called a $n$-**tensor**.

**Proposition 5.2** (Universal Property). *We have a multi $\mathbb{R}$-linear map $q : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$, $(v_1, v_2, \cdots, v_n) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$. It satisfies the universal property:*

$\forall$ *multi $\mathbb{R}$-linear map $f : V_1 \times \cdots \times V_n \to W$, $\exists$ unique linear map $\widetilde{f} : V_1 \otimes \cdots \otimes V_n \to W$ s.t. $\widetilde{f} \circ q = f$. i.e. The diagram commutes:*

$$\begin{array}{ccc} V_1 \otimes \cdots \otimes V_n & & \\ \rho \big\uparrow & \searrow^{\exists! \widetilde{f}} & \\ V_1 \times \cdots \times V_n & \xrightarrow{f} & W \end{array}$$

**Corollary 5.3.**

$$\begin{gathered} \{\textit{multi } \mathbb{R}\textit{-linear maps } V_1 \times \cdots \times V_n \to W\} \cong \{\textit{linear map } V_1 \otimes \cdots \otimes V_n \to W\} \\ f \leftrightarrow \widetilde{f} \end{gathered} \tag{5.5}$$

**Proposition 5.4.**

- *Any element in $V_1 \otimes \cdots \otimes V_n$ can be written as $\sum a_i v_i^1 \otimes \cdots \otimes v_i^n$ for some $a_i \in \mathbb{R}$.*
- *If $(e_i^j)_{j \in \mathcal{A}_i}$ is a basis for $V_i$, then $\{e_1^{j_1} \otimes e_2^{j_2} \otimes \cdots \otimes e_n^{j_n} | j_i \in \mathcal{A}_i\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$.*
- $\dim(V_1 \otimes \cdots \otimes V_n) = \prod_{i=1}^{n} \dim(V_i)$

**Proposition 5.5.** *Denote* $W^* = \mathrm{Hom}(W, \mathbb{R})$*, then we have an injection*

$$\begin{aligned} V \otimes W^* &\xrightarrow{e} \mathrm{Hom}(W, V) \\ v \otimes f &\mapsto (w \mapsto f(w) \cdot v) \end{aligned} \tag{5.6}$$

*If* $\dim V$ *or* $\dim W$ *is finite, then* $e$ *is an isomorphism.*

*Indeed, if* $\dim V = \infty$*, then* $\mathrm{id}_V \notin e(V \otimes V^*)$

Given any $l_i \in \mathrm{Hom}(V_i, W_i)$, $1 \leqslant i \leqslant n$, we define

$$\begin{aligned} l_1 \otimes \cdots \otimes l_n &\in \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots W_n) \\ (l_1 \otimes \cdots \otimes l_n)(v_1 \otimes \cdots \otimes v_n) &= l_1(v_1) \otimes \cdots \otimes l_n(v_n) \end{aligned} \tag{5.7}$$

**Proposition 5.6.** *If* $\dim V_i < \infty$*,* $\forall 1 \leqslant i \leqslant n$*, then we have isomorphism*

$$\begin{aligned} V_1^* \otimes \cdots \otimes V_n^* &\xrightarrow{\cong} (V_1 \otimes \cdots \otimes V_n)^* \\ f_1 \otimes \cdots \otimes f_n &\mapsto \left( (v_1 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^n f_i(v_i)) \right) \end{aligned} \tag{5.8}$$

For $\bigotimes_n V = \underbrace{V \otimes \cdots \otimes V}_{n}$, $S_n = \{\text{bijection on } \{1, 2, \cdots, n\}\}$ acts on $\bigotimes_n V$, where

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{5.9}$$

A tensor $T \in \bigotimes_n V$ is called **symmetric** if $\sigma(T) = T$, $\forall \sigma \in S_n$.

$T$ is called **anti-symmetric** if $\sigma(T) = \mathrm{sgn}(\sigma) \cdot T$, $\forall \sigma \in S_n$.

Define

$$\begin{aligned} \mathrm{Sym}^n(V) &= \{\text{symmetric tensors in } \bigotimes_n V\} \\ \bigwedge^n(V) &= \{\text{anti-symmetric tensors in } \bigotimes_n V\} \end{aligned} \tag{5.10}$$

which are both in $\bigotimes_n V$. And

$$\dim(\mathrm{Sym}^n(V)) = \binom{\dim(V)+n-1}{n} \quad \dim(\bigwedge{}^n V) = \binom{\dim(V)}{n} \tag{5.11}$$

From now on, **we may assume** $\dim V < \infty$. Define

$$L^n(V) = \left(\bigotimes_n V\right)^* \cong \bigotimes_n V^* \cong \{\text{multi } \mathbb{R}\text{-linear maps } V_1 \times \cdots \times V \to \mathbb{R}\} \tag{5.12}$$

And by the assumption we can obtain

$$\begin{aligned} \mathrm{Sym}^n(V^*) &\cong \{\text{symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \\ \bigwedge{}^n(V^*) &\cong \{\text{anti-symmetric multi } \mathbb{R}\text{-linear maps } l : V \times \cdots \times V \to \mathbb{R}\} \end{aligned} \tag{5.13}$$

We will mainly focus on $\bigwedge^n(V^*)$, also denoted as $\mathrm{Alt}^k(V)$= $\bigwedge^n(V^*)$. An element in $\mathrm{Alt}^k(V)$ is called a (linear) $k$-**form** on $V$ Now for $V = \mathbb{R}\langle e_1, \cdots, e_n\rangle$, $V^* = \mathbb{R}\langle e_1^*, \cdots, e_n^*\rangle$. Then

$$L^2(V) = \{\text{all bilinear forms on } V\}$$
$$L^2(V) \cong \mathrm{Sym}^2(V^*) \oplus \bigwedge{}^2(V^*)$$

And $\mathrm{Sym}^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* + e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is symmetric bilinear form

$\mathrm{Alt}^2(V) = \bigwedge^2(V^*) = \mathbb{R}\left\langle e_i^* \otimes e_j^* - e_j^* \otimes e_i^* | 1 \leqslant i \leqslant i \leqslant n\right\rangle$ is anti-symmetric bilinear form.

The determinant $\det \in \mathrm{Alt}^n(\mathbb{R}^n)$.

**Definition 5.7** (Exterior product)**.**

$$\bigwedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^l(V) \to \mathrm{Alt}^{k+l}(V)$$

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$
$$= \sum_{\sigma \in S_{k,l}} \mathrm{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)})\omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

where $S_{k,l} = \{\sigma \in S_{k+l} | \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\} \subset S_{k+l}$.

Then we have those properties:

**Proposition 5.8.**

*(1)* $\omega_1 \wedge \omega_2 = (-1)^{|\omega_1|\cdot|\omega_2|}\omega_2 \wedge \omega_1$, $|\omega| = k$ *is* $\omega \in \mathrm{Alt}^k(V)$. *In particular,* $\omega \wedge \omega = 0$ *if* $|\omega|$ *is odd.*

*(2)* $(\omega_1 \wedge \omega_2) \wedge w_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

*(3) Given any* $\omega_1, \cdots, \omega_k \in \mathrm{Alt}^1(V) = V^*$, $v_1, \cdots, v_k \in V$. *Then*

$$(\omega_1 \wedge \cdots \wedge \omega_k)(v_1, \cdots, v_k) = \det\left[w_i(v_j)\right]_{i,j} \tag{5.14}$$

*Moreover,* $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$ *iff* $\omega_i$ *are linearly independent.*

*(4)* $V = \mathbb{R}\langle e_1, \cdots, e_n\rangle$. *Then*

$$\mathrm{Alt}^k(V) = \mathbb{R}\left\langle e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} | i_1 < \cdots < i_k\right\rangle \tag{5.15}$$

*In particular,* $\mathrm{Alt}^n(V) = \mathbb{R}\langle e^*_1 \wedge \cdots \wedge e^*_n\rangle$. *And we denote* $\mathrm{Alt}^0(V) = \mathbb{R}$, $\mathrm{Alt}^k(V) = 0$, $k > n$.

*(5) Any* $f \in \mathrm{Hom}(V, W)$ *induces* $\mathrm{Alt}^k(f) \in \mathrm{Hom}(\mathrm{Alt}^k(V), \mathrm{Alt}^k(W))$, *where*

$$\mathrm{Alt}^k(f)(\omega)(w_1, \cdots, w_k) = \omega(f(w_1), \cdots, f(w_k)) \tag{5.16}$$

*We have* $\mathrm{Alt}^k(f \circ g) = \mathrm{Alt}^k(g) \circ \mathrm{Alt}^k(f)$, $\mathrm{Alt}^k(\mathrm{id}_V) = \mathrm{id}_{\mathrm{Alt}^k(V)}$. *Such* $\mathrm{Alt}^k(-)$ *is called a* ***contravariant functor****.*

*Proof.*

(1) By definition,

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) = \omega_2 \wedge \omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k+l)})$$

where $\sigma(i) = \begin{cases} i+k & 1 \leqslant i \leqslant l \\ i-l & l+1 \leqslant i \leqslant k+l \end{cases}$. $\mathrm{sgn}(\sigma) = (-1)^{k+l}$.

(2) By definition.

(3) By linearity, we assume $\omega_i = e^*_{a(i)}, v_j = e_{b(j)}$ for some $a(i), b(j)$. Further more, can assume $\{a(i)\} = \{b(i)\}$. (Otherwise, $LHS = RHS = 0$.)

Then $e^*_{a(i)}(e_{b(j)}) = \delta_{a(i),b(j)}$. After permutation, may assume $a(i) = b(i), \forall i$. It is direct to check $LHS = 1 = RHS$.

(4) If $\omega_1, \cdots, \omega_k$ are linear independent. Then $\exists$ basis $e^*_1, \cdots, e^*_n$ of $V^*$, basis $e_1, \cdots, e_n$ of $V$ *s.t.* $\omega_i = e^*_i$, $\forall 1 \leqslant i \leqslant n$.

$$(\omega_1 \wedge \cdots \wedge \omega_n)(e_1, \cdots, e_n) = \det(I) = 1 \neq 0 \Rightarrow \omega_1 \wedge \cdots \wedge \omega_n \neq 0$$

If $\omega_1, \cdots, \omega_k$ are linearly dependent. WLOG, we assume $\omega_k = \sum_{i=1}^{k-1} a_i \omega_i$.

$$(\omega_1 \wedge \cdots \wedge \omega_k)(e_1, \cdots, e_n) = \sum_{i=1}^{k-1} a_i (\omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_i)(e_1, \cdots, e_n) = 0$$

(5) For $i_1 < \cdots < i_k, j_1 < \cdots < j_n$ we have

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1 & j_t = i_t, \forall 1 \leqslant t \leqslant k \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

Since $\dim \operatorname{Alt}(V) = \dim \bigwedge^k(V^*) = \binom{n}{k} = |\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k\}|$.

(6) For $\omega \in \operatorname{Alt}^k(W), f \in \operatorname{Hom}(V, W)$, define $\operatorname{Alt}^k(f)(\omega) \in \operatorname{Alt}^k(V)$ by

$$\operatorname{Alt}^k(f)(\omega(V_1, \cdots, V_k)) = \omega(f_* V_1, \cdots, f_* V_k) \in \mathbb{R}$$

□

**Definition 5.9.**

An **$\mathbb{R}$-algebra** consists of an $\mathbb{R}$-vector space $A$ with a bilinear map $\mu : A \times A \to A$ that is associate, *i.e.* $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$.

Say $A$ is **unitary** if $\exists 1 \in A$ *s.t.* $\mu(a, 1) = \mu(1, a) = a, \forall a \in A$

Say $A$ is **graded** if $A = \bigoplus_{k \in \mathbb{Z}} A_k$ as vector space, and $\mu(A_k \times A_l) \subset A_{k+l}$. Elements in $A_k$ are called **homogeneous elements** of degree $k$.

If $A$ is graded $\mathbb{R}$-algebra, we say $A$ is **anticommutative** if $\mu(a, b) = (-1)^{k+l}\mu(b, a), \forall a \in A_k, b \in A_l$. And say $A$ is **commutative** if $\mu(a, b) = \mu(b, a), \forall a, b$.

If $A$ is graded $\mathbb{R}$-algebra, say $A$ is **connected** if $\exists$ unit $1 \in A_0$ *s.t.* the map $\varepsilon : \mathbb{R} \to A_0, r \mapsto r \cdot 1$ is an isomorphism.

Given vector space $V$, let

$$\begin{array}{ccc} \operatorname{Alt}^k(V) & = & \bigoplus_{k \geqslant 0} \operatorname{Alt}^k(V) \\ \| & & \| \\ \operatorname{Alt}^*(V^*) & = & \bigoplus_{k \geqslant 0} \wedge^k(V^*) \end{array}$$

By Proposition 5.8, we have the theorem

**Theorem 5.10.** $(\mathrm{Alt}^*(V), \wedge)$ *is a graded connected anticommutative* $\mathbb{R}$*-algebra, called the* ***exterior algebra*** *of* $V$ *or* ***exterior algebra*** *of* $V$

## 5.3 Operation on Vector Bundles

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, meaning a vector bundle $E \xrightarrow{\pi} M$ of dimension $n$, local trivialization $\left\{U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n\right\}_{\alpha \in \mathcal{A}}$. By shrinking $U_\alpha$, we may assume we have an smooth atlas $\left\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^m\right\}_{\alpha \in \mathcal{A}}$.

For $x \in M$, use $E_x$ to denote $\pi^{-1}(x)$, fiber over $x$, which is a vector space of dimension $n$.

Then **Dual bundle of a vector bundle** $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$ is

$$E^* := \{(x, l) | x \in M, l \in (E_x)^*\}, \pi' : E^* \to M, (x, l) \mapsto x, (\pi')^{-1}(x) = (E_x)^* \quad (5.18)$$

Define topology or smooth structure on $E^*$ *s.t.* $\pi' : E^* \to M$ is a smooth vector bundle.

For $\alpha \in \mathcal{A}$, let $E^*_\alpha = \pi'^{-1}(U_\alpha)$, we have a bijection

$$\widetilde{\varphi}_\alpha : E^*_\alpha \xrightarrow{bijection} \mathbb{R}^m \times (\mathbb{R}^n)^* \xrightarrow{\quad\cong\quad} \mathbb{R}^{m+n}$$

$$(x, l) \longmapsto (\psi_\alpha(x), (\varphi_{\alpha,x})^{-1}(l))$$

We can check that

(1) $\{\widetilde{\varphi}_\alpha^{-1} | \alpha \in \mathcal{A}, V \subset \mathbb{R}^{m+n} \text{ open}\}$ is a basis, we use it to generate a topology on $E^*$.

(2) Use $\widetilde{\varphi}_\alpha : E^*_\alpha \xrightarrow{\cong} \mathbb{R}^{m+n}, \alpha \in \mathcal{A}$ as an atlas to give $E^*$ a smooth structure.

(3) $E^* \xrightarrow{\pi'} M$ is a smooth vector bundle, called the **dual vector bundle** of $E \xrightarrow{\pi} M$, where

$$(E^*)_x = E_x^*$$

We can define other operations on vector bundles in similar way:

Given $\mathbb{R}^n \hookrightarrow E \xrightarrow{\pi} M$, $\mathbb{R}^m \hookrightarrow F \xrightarrow{\pi} M$, we can define

$$\mathbb{R}^{m+n} \hookrightarrow E \oplus F \xrightarrow{\pi} M \text{ with } (E \oplus F)_x = E_x \oplus F_x$$

$$\mathbb{R}^{mn} \hookrightarrow E \otimes F \xrightarrow{\pi} M \text{ with } (E \otimes F)_x = E_x \otimes F_x$$

$$\mathbb{R}^{mn} \hookrightarrow \mathrm{Hom}(E, F) \xrightarrow{\pi} M \text{ with} \mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x)$$

$$\mathbb{R}^{\binom{n}{k}} \hookrightarrow \mathrm{Alt}^k(E) \to M \text{ with}$$

$$\mathrm{Alt}^k(E)_x = \mathrm{Alt}^k(E_l) = \{\text{alternating } k\text{-linear } l : E_x \times \cdots \times E_x \to \mathbb{R}\}$$

Then $\mathrm{Alt}^k(TM) = \bigwedge^k(T^*M)$.

$$\mathrm{Alt}^k(M)_x = \{\text{alternating } k\text{-linear } l : T_xM \times \cdots \times T_xM \to \mathbb{R}\} = \{\text{linear } k\text{-form on } T_xM\}$$

Define

$$\Gamma(E) := \{\text{smooth sections of } E\} = \{s \in C^\infty(M, E) : \pi \circ s = \mathrm{id}_M\}$$

**Definition 5.11.** Given smooth $M$, define a differential $k$-form on $M$ to be an element in $\Gamma(\mathrm{Alt}^k(TM))$ is a differential $k$-form $\alpha$ assigns each $x \in M$ a linear $k$-form $\alpha(x) \in \mathrm{Alt}^k(T_xM)$.

Denote $\Omega^k(M)$ be the set of all the differential $k$-forms.

Then $\Omega^0(M) = C^\infty(M, \mathbb{R})$. $\mathrm{Alt}^1(TM) = T^*M \Rightarrow$ a 1-form on $M$ is just a "cotangent vector field" on $M$.

$\Omega^k(M) = 0$ if $k \geqslant \dim(M)$.

## 5.4 Differential forms using local chart

Given local chart $(U, x^1, \cdots, x^n)$ of $M$.

For any $p \in U$, $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leqslant i \leqslant n}$ is a basis of $T_x M$.

We denote the dual basis of $T_x^* M$ by $\{\mathrm{d}x^i|_p\}_{1 \leqslant i \leqslant n}$.

For any $\alpha \in \Omega^1(M)$, $\alpha|_U$ can be written as $\sum_{i=1}^n f_1 \mathrm{d}x^i$, where $f^i \in C^\infty(U, \mathbb{R})$.

Similarly, $\{\mathrm{d}x^{i_1}|_1 \wedge \cdots \wedge \mathrm{d}x^{i_k}|_p | i_1 < \cdots < i_k\}$ is a basis for $\bigwedge^k(T_x^* M)$, so $\forall \alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_{i_1 < \cdots < i_k} f_{i_1, \cdots, i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},\ f_{i_1, \cdots, i_k} \in C^\infty(U, \mathbb{R})$$

We give the notation that $I = (i_1, \cdots, i_k)$, write $f_{i_1, \cdots, i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ as $f^I \mathrm{d}x^I$.

**Change of coordinate** If $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^n)$ two charts of $M$ and $p \in U \cap V$, then

$$\mathrm{d}y^i = \sum_{1 \leqslant i \leqslant n} \frac{\partial y_i}{\partial x^i} \mathrm{d}x^i. \tag{5.19}$$

## 5.5 Exterior Differential

For $k = 0$, define $d : \Omega^0(M) \to \Omega^1(M)$ as follows:

$\forall p \in M$, $X_p \in T_p M$, $\mathrm{d}f|_p(X_p) = X_p(f) \in \mathbb{R}$. In local chart, $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x^i$.

**Theorem 5.12.** $\exists$ *linear operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ *s.t. For* $\alpha \in \Omega^k(M)$,

$$\alpha|_U = \sum_I f^I \mathrm{d}x^I \Rightarrow \mathrm{d}\alpha|_U = \sum_I \mathrm{d}f^I \wedge \mathrm{d}x^I \tag{5.20}$$

*Called the **exterior differential***

*Proof.* It suffices to prove that (5.20) is compatible for two charts $(U, x^1, \cdots, x^n), (V, y^1, \cdots, y^n)$, *i.e.* the diagram is commutative.

$$\begin{array}{ccc}
f\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^k & \longleftrightarrow & \displaystyle\sum_{1 \leqslant i_1, i_2, \cdots, i_k \leqslant n} f \frac{\partial y^{i_1}}{\partial x^1} \cdots \frac{\partial y^{i_k}}{\partial x^k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \\
\Big\downarrow \mathrm{d} & & ?\Big\downarrow d \\
\displaystyle\sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial y^i} \mathrm{d}y^i \wedge \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k} & \overset{?}{\longleftrightarrow} & \displaystyle\sum_{1 \leqslant i_1, i_2, \cdots, i_k \leqslant n} \frac{\partial f}{\partial x^j} \cdot \frac{\partial y^{i_1}}{\partial x^1} \cdots \frac{\partial y^{i_k}}{\partial x^k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}
\end{array}$$

□

**Theorem 5.13.**

*(1)* $\mathrm{d}^2 = 0$.

*(2)* $\forall \alpha \in \Omega^k(M), \beta \in \Omega^l(M)$, $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$.

*Proof.*

(1) If $\alpha|_U = \sum_U f^I \mathrm{d}x^I$. By linearity suffices to check

$$\begin{aligned}
\mathrm{d} \circ \mathrm{d}(f\mathrm{d}x^I) &= \mathrm{d}\left( \sum_{1 \leqslant i \leqslant n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \right) \\
&= \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \frac{\partial^2 f}{\partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I \\
&= 0
\end{aligned}$$

(2) By linearity, suffices to assume $\alpha = f\mathrm{d}x^I$, $\beta = g\mathrm{d}x^I$.

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= \mathrm{d}(fg\mathrm{d}x^I \wedge x^J) \\
&= \sum_{1 \leqslant i \leqslant n} \frac{\partial (fg)}{\partial x^i} \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \sum_{1 \leqslant i \leqslant n} \left( \frac{f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J
\end{aligned}$$

And

$$\mathrm{d}\alpha \wedge \beta = \sum_i \frac{\partial f}{\partial x^i} g \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J$$

$$\alpha \wedge \mathrm{d}\beta = \sum_i \frac{\partial g}{\partial x^i} f \mathrm{d}x^I \wedge \mathrm{d}x^i \wedge \mathrm{d}x^J = \sum_i (-1)^k \frac{\partial g}{\partial x^i} f \mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J$$

□

**Example 5.14.** For $M = \mathbb{R}^3$,

$$\begin{array}{ccccc}
\Omega^0(\mathbb{R}^3) & = & C^\infty(\mathbb{R}^3) & & \\
\downarrow d & & \downarrow \text{gradient} & & \\
\Omega^1(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{X}\mathbb{R}^3, & & f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z \longleftrightarrow f\partial x + g\partial y + h\partial z \\
\downarrow d & & \downarrow \text{curl} & & \\
\Omega^2(\mathbb{R}^3) & \longleftrightarrow & \mathfrak{X}\mathbb{R}^3, & & f\mathrm{d}x \wedge \mathrm{d}y + g\mathrm{d}x \wedge \mathrm{d}z + h\mathrm{d}y \wedge \mathrm{d}z \longleftrightarrow f\partial z + g\partial x + h\partial y \\
\downarrow d & & \downarrow \text{divergent} & & \\
\Omega^3(\mathbb{R}^3) & \longleftrightarrow & C^\infty(\mathbb{R}^3), & & f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \longleftrightarrow f
\end{array}$$

## 5.6 Pull Back of Differential Forms

For $F \in C^\infty(M, N)$, $\alpha \in \Omega^k(N)$, define the **pullback** $F^*(\alpha) \in \Omega^k(M)$ as follows:

$$\forall p \in M, V_1 \cdots, V_k \in T_pM, F^*(\alpha)|_p(V_1, \cdots, V_k) = \alpha|_{F(p)}(F_{p,*}(V_1), \cdots, F_{p,*}(V_k)) \in \mathbb{R}$$

Actually, $F^*|_p = \mathrm{Alt}^k(F_{p,*}) : \mathrm{Alt}^k(T_{F(p)N}) \to \mathrm{Alt}^k(T_pM)$.

**Proposition 5.15.** *For $F : M \to N, G : N \to L$.*

*(1)* $f \in \Omega^0(N)$, $F^*(f) = f \circ F \in \Omega^0(M)$.

*(2)* $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$.

*(3)* $F^*(\mathrm{d}\alpha) = \mathrm{d}F^*(\alpha)$.

*(4)* $(G \circ F)^* = F^* \circ G^*$

*Proof.*

(1)
$$\begin{array}{ccc} \mathrm{Alt}^0(T_{F(p)}N) & \xrightarrow{\mathrm{Alt}^0(F_{p,*})} & \mathrm{Alt}^0(T_pM) \\ \Vert & & \Vert \\ \mathbb{R} & \xrightarrow{\quad id \quad} & \mathbb{R} \end{array} \quad \text{commutes.}$$

(2)
$$\begin{array}{ccc} \mathrm{Alt}^k(T_{F(p)N}) \times \mathrm{Alt}^l(T_{F(p)}N) & \xrightarrow{\ \wedge\ } & \mathrm{Alt}^{k+l}(T_pM) \\ \big\downarrow{\scriptstyle \mathrm{Alt}^k(F_{p,*})\times \mathrm{Alt}^l(F_{p,*})} & & \big\downarrow{\scriptstyle \mathrm{Alt}^{k+l}(F_{p,*})} \\ \mathrm{Alt}^k(T_pM) \times \mathrm{Alt}^l(T_pM) & \xrightarrow{\ \wedge\ } & \mathrm{Alt}^{k+l}(T_pM) \end{array} \quad \text{commutes.}$$

(3) By linearity it suffices to check

$$\mathrm{d}F^*(f\mathrm{d}x^I) = F^*\mathrm{d}(f\mathrm{d}x^I)$$

By Leibniz rule for $\mathrm{d}$ and (2), it suffices to show

(a) $\mathrm{d}F^*(\mathrm{d}f) = F^*(\mathrm{d}f)$

(b) $\mathrm{d}F^*(\mathrm{d}x^i) = F^*(\mathrm{d}(\mathrm{d}x^i))$

Which leaves to the readers.

(4) By definition.

□

**Definition 5.16.** A $k$-form $\omega$ is **closed** if $\omega \in \ker\left(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)\right)$.

A $k$-form $\omega$ is **exact** if there exists a $(k-1)$-form $\eta$ such that $d\eta = \omega$, or equivalently, $\omega \in \mathrm{Im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right)$.

By Proposition 5.15 (1), exact $k$-form are all closed.

So we may define the $k$-th **de Rham cohomology** of $M$

$$H^k_{\mathrm{DR}}(M) := \frac{\ker\left(d : \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)\right)}{\mathrm{Im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right)} \tag{5.21}$$

By Proposition 5.15 (2), we have $\forall F \in C^\infty(M,N)$, $\omega \in \Omega^k(N)$.

Then $\omega$ closed $\Rightarrow F^*\omega$ is closed. $\omega$ exact $\Rightarrow F^*\omega$ exact.

So $F$ induces a linear map

$$\begin{aligned} F^* : H^k_{\mathrm{DR}}(N) &\to H^k_{\mathrm{DR}}(M) \\ [\omega] &\mapsto [F^*\omega] \end{aligned}$$

**Proposition 5.17** (Key properties of $H^k_{\mathrm{DR}}(M)$)**.**

*(1)* $(F \circ G)^* = G^* \circ F^*$

*(2)* $(\mathrm{id})^* = \mathrm{id}$.

*(3)* $F, G \in C^\infty(M,N)$, *$F$ homotopic to $G$* $\Rightarrow F^* = G^*$

*(4)* *If $F$ is a homotopy equivalence* $\Rightarrow F^* : H^k_{\mathrm{DR}}(M) \to H^k_{\mathrm{DR}}(N)$ *is an isomorphism.*

**Remark 5.18.** Properties (3),(4) are nontrivial, which is the essential part of the theory of de Rham cohomology

**Proposition 5.19.** $H^0_{\mathrm{DR}}(M) \cong \mathbb{R}\langle \pi_0(M)\rangle$, *where* $\pi_0(M) = \{$*path component of* $M\}$.

It suffices to prove the lemma that

**Lemma 5.20.** $\alpha \in \Omega^0(M) = C^\infty(M,\mathbb{R}^n)$. *Then $\alpha$ is closed iff $\alpha$ is constant on each component of $M$.*

*Proof.* The inverse part is trivial.

Assume $\alpha$ is closed. Pick $p, q \in M$ in some path component. $\exists$ smooth path $\gamma : \mathbb{R} \to M$, $\gamma(0) = p, \gamma(1) = q$.

$\mathrm{d}\alpha = 0 \Rightarrow \mathrm{d}(\gamma^*\alpha) = 0 \Rightarrow \mathrm{d}(\alpha \circ \gamma) = 0 \Rightarrow \dfrac{\mathrm{d}(\alpha\circ\gamma)}{\mathrm{d}t} = 0 \Rightarrow \alpha\circ\gamma(1) = \alpha\circ\gamma(0)$.

So $\alpha(p) = \alpha(q)$ ∎

We have $H^k_{\mathrm{DR}}(M) \cong \mathrm{Ab}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{R}$. $\mathrm{Ab}(\pi_1(M))$ is the Abelian group of $\pi_1(M)$. In particular, $H^1_{\mathrm{DR}}(\mathbb{R}^2) = 0$, $H^1_{\mathrm{DR}}(\mathbb{R}^2\backslash\{0\}) \neq 0$.

Let us stop the discussion of de Rham cohomology for a moment, and move on to the next topic.

# 6 Orientation and Integration of Differential Form

## 6.1 Orientation on Manifold

An orientation on a finite dimensional vector space $V$ is an equivalent class of ordered basis

$$\alpha = (\alpha_1, \cdots, \alpha_n)^T \sim \beta = (\beta_1, \cdots, \beta_n)^T \Leftrightarrow \det(\alpha\beta^T) > 0$$

Each vector space has exactly two orientations. And we actually have the 1-1 correspondence

$$\{\text{orientation on } V\} \leftrightarrow (\mathrm{Alt}^n(V)\backslash\{0\})/_{\mathbb{R}^+}$$

$$[(e_1, \cdots, e_n)] \leftrightarrow [e_1^* \wedge \cdots \wedge e_n^*]$$

An **orientation form** on $M$ of dimension $n$ is a nowhere vanishing $\omega \in \Omega^n(M)$ *i.e.* an orientation form is a nowhere vanishing section of $\mathrm{Alt}^n(TM)$.

Two orientation forms $\omega_1, \omega_2$ are equivalent if $\exists f \in C^\infty(M, \mathbb{R}^+)$ *s.t.* $\omega_1 = f\omega_2$. An **orientation** on $M$ is an equivalent class of orientation form.

An **orientation manifold** is a manifold that has an orientation.

An **oriented manifold** is a manifold equipped with an orientation.

**Example 6.1.** $|\pi_0(M)| = k \Rightarrow M$ has $2^k$ orientations or no orientations.

**Example 6.2.**

1. $U \subset \mathbb{R}^n$ open. $U$ has a standard orientation, represented by the form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Denote this standard orientation as $\mathcal{O}_{\mathrm{std}}$

2. $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ oriented manifolds $\Rightarrow (M \times N, \mathcal{O}_M \times \mathcal{O}_N)$. If $\mathcal{O}_M = [\omega_M], \mathcal{O}_N = [\omega_N]$, then $\mathcal{O}_M \times \mathcal{O}_N$ is defined by $[\pi_M^*(\omega_M) \wedge \pi_N^*(\omega_N)]$. ($\pi_M, \pi_N$ is the pullback of the projection map)

3. $T^n, S^n$ are orientable.

4. $\mathbb{RP}^n$ orientable iff $n$ is odd.

**Proposition 6.3.** *Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of $M$. Suppose we have an orientation $\mathcal{O}_\alpha$ on each $U_\alpha$ s.t. $\mathcal{O}_\alpha|_{U_\alpha \cap U_\beta} = \mathcal{O}_\beta|_{U_\alpha \cap U_\beta}$, $\forall \alpha, \beta$. Then $\exists$ unique orientation $\mathcal{O}_M$ on $M$ s.t. $\mathcal{O}_M|_{U_\alpha} = \mathcal{O}_\alpha$.*

*Proof.* For each $\alpha$, we have $\omega_\alpha \in \Omega^n(U_\alpha)$ nowhere-vanishing. And

$$\omega_\alpha|_{U_\alpha \cap U_\beta} = f_{\alpha\beta} \cdot \omega_\beta|_{U_\alpha \cap U_\beta},\ f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^+ \tag{6.1}$$

Take partition of unity subordinate to $\mathcal{U}$, $\{\varphi_\alpha\}$.

Set $\omega = \sum_\alpha \varphi_\alpha \cdot \omega_\alpha$. Then $\omega$ is nowhere-vanishing by (6.1).

The uniqueness follows from the fact that $n$-form is equivalent if and only if it is equivalent on each chart. □

**Definition 6.4.** Given $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$. $f \in \mathrm{Diff}(M, N)$. Say $f$ is **orientation preserving** if $f^*(\mathcal{O}_N) = \mathcal{O}_M$. $f$ is **orientation reversing** if $f^*(\mathcal{O}_N) = -\mathcal{O}_M$.

**Lemma 6.5.** *$U_1, U_2 \subset \mathbb{R}^n$ open. Then $f : (U_1, \mathcal{O}_{\mathrm{std}}) \to (U_2, \mathcal{O}_{\mathrm{std}})$ is orientation preserving iff*

$$\forall p \in U_1, \det(\mathrm{D}f|_p) > 0 \quad \mathrm{D}f = \left(\frac{\partial f^i}{\partial x^j}\right)_{1 \leqslant i,j \leqslant n}$$

*Proof.* For $\mathcal{O}_{\text{std}} = [\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n]$,

$$f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^n, \quad \mathrm{d}f^i = \sum_{i=1}^{n} \frac{\partial f^i}{\partial x^i}\mathrm{d}x^i$$
$$= \det(\mathrm{D}f)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

Then

$$\begin{aligned}\det(\mathrm{D}f) > 0 &\Leftrightarrow f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \sim \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &\Leftrightarrow f^*\mathcal{O}_{\text{std}} = \mathcal{O}_{\text{std}} \\ &\Leftrightarrow f \text{ is orientation preserving}\end{aligned}$$

□

Given $(M, \mathcal{O})$, $p \in M$, a basis $e_1, \cdots, e_n$ of $T_pM$ is called **oriented** if $\mathcal{O}_p = [(e_1, \cdots, e_n)]$.

A chart $U \xrightarrow{\varphi} V \overset{\text{open}}{\subset} \mathbb{R}^n$ is **oriented** if $\varphi^*(\mathcal{O}_{\text{std}}) = \mathcal{O}|_U$.

A smooth atlas $\left\{U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha\right\}_{\alpha \in \mathcal{A}}$ is **oriented** if each chart $U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha$ is oriented.

A smooth atlas $\left\{U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha\right\}_{\alpha \in \mathcal{A}}$ is called **positive** if $\forall \alpha, \beta \in \mathcal{A}$,

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha' : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \text{ is orientation preserving}$$

By Lemma 6.5, this is equivalent to $\det(\mathrm{D}\varphi_{\alpha\beta}|_p) > 0$ for any $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

## 6.2 Integration on Oriented Manifold

**Goal:** Given $M, \mathcal{O}$, $\omega \in \Omega_c^n(M) = \{\text{compactly supported } n\text{-form on } M\}$. Then $\text{Supp}(\omega) = \overline{\{p \in M | \omega_p \neq 0 \in \text{Alt}^n(T_pM)\}}$ is compact.

We hope to define $\int_M \omega \in \mathbb{R}$.

For $M \overset{\text{open}}{\subset} \mathbb{R}^n$, $\mathcal{O} = \mathcal{O}_{\text{std}}$. Then $\forall \omega \in \Omega_c^n(M)$,

$$\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,\ f \in C_c^\infty(M)$$

Define $\int_M \omega = \int_M f \mathrm{d}\mu$ where $\mu$ is the standard Lebesgue measure on $\mathbb{R}^n$.

**Lemma 6.6.** *$U, V \overset{open}{\subset} \mathbb{R}^n$, $\varphi : U \xrightarrow{\cong} V$ is orientation preserving. Then $\forall \omega \in \Omega_c^n(V)$, we have $\int_U \varphi^*(\omega) = \int_V \omega$.*

*Proof.* If $\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, then

$$\begin{aligned} \varphi^*(\omega) &= \varphi^*(f) \wedge \mathrm{d}\varphi^1 \wedge \cdots \wedge \mathrm{d}\varphi^n \\ &= (f \circ \varphi) \det \left( \frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leqslant i,j \leqslant n} \mathrm{d}x^{1^1} \wedge \cdots \wedge \mathrm{d}x^{1^n} \end{aligned} \tag{6.2}$$

So $\int_U \varphi^*(\omega) = \displaystyle\int_U (f \circ \varphi) \det \left( \frac{\partial \varphi^i}{\partial x^j} \right)_{1 \leqslant i,j \leqslant n} \mathrm{d}\mu = \int_V f \mathrm{d}\mu = \int_V \omega$

□

So we can define the integral over special $\omega$ and general $M$.

**Definition 6.7.** If $\omega \in \Omega_s^n(M) = \{n\text{-forms with "small" support}\}$
$= \{\omega \in \Omega_c^n(M) | \exists \text{oriented chart } \varphi : U \xrightarrow{\cong} V \quad \textit{s.t.}\ \mathrm{Supp}(\omega) \subset U\}$.

We define $\int_M \omega := \int_V \varphi^{-1,*}(\omega)$

*Claim.* If $\mathrm{Supp}(\omega) \subset U_\alpha \cap U_\beta$, then $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$

*Proof.*

$$\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\cong} \varphi_\beta(U_\alpha \cap U_\beta)$$
$$\varphi_\alpha^{-1,*}(\omega) \mapsto \varphi_\beta^*(\omega)$$

By Lemma 6.6, we have $\int_{V_\alpha} \varphi_\alpha^{-1,*}(\omega) = \int_{V_\beta} \varphi_\beta^{-1,*}(\omega)$ □

**Theorem 6.8.** *For any oriented $(M, \mathcal{O})$, $\exists$ unique linear map $\int_M : \Omega_c^n(M) \to \mathbb{R}$ that extends $\int_M : \Omega_s^n(M) \to \mathbb{R}$.*

*Proof.*

**Step1:** There exists an oriented atlas $\mathcal{U} = \{\varphi_\alpha : U_\alpha \to V_\alpha \overset{\text{open}}{\subset} \mathbb{R}^n\}_{\alpha \in \mathcal{A}}$. Indeed, pick any smooth atlas $\mathcal{U} = \{U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha\}_{\alpha \in \mathcal{A}}$. By replacing $\varphi_\alpha$ with $r \circ \varphi_\alpha$ where $r(x_1, \cdots, x_n) = (-x_1, \cdots, x_n)$. We can get the oriented atlas $\mathcal{U}'$.

**Step2.** Pick a partition of unity subordinate to $\mathcal{U}$, $\{\varphi_\alpha : M \to [0, 1]\}$

Now we begin the main proof:

Let $\omega_\alpha = \rho_\alpha \cdot \omega$. $\text{Supp}(\omega_\alpha) \subset \text{Supp}(\rho_\alpha) \cap \text{Supp}(\omega) \subset U_\alpha$. And $\omega_\alpha \in \Omega_s^n(M)$

*Claim.* $\omega_\alpha \neq= 0$ for only finite many $\alpha \in \mathcal{A}$

*Proof.* $\forall p \in \text{Supp}(\omega)$, $\exists$ neighbourhood $W_p$ only intersects $\text{Supp}(\rho_\alpha)$ for finitely many $\alpha$.

Since $\{W_p\}_{p \in \text{Supp}(\omega)}$ is an open cover of $\text{Supp}(\omega)$, by compactness, $\text{Supp}(\omega)$ only intersects $\text{Supp}(\rho_\alpha)$ for finitely many $\alpha$.

Therefore, $\omega_\alpha \neq 0$ for only finitely many $\alpha$ □

By this claim, since $\sum_{\alpha \in \mathcal{A}} \rho_\alpha = 1 \Rightarrow \omega = \omega_{\alpha_1} + \cdots + \omega_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_k \in \mathcal{A}$. We may define

$$\int_M \omega = \sum_{\alpha \in \mathcal{A}} \int_M \omega_\alpha = \int_M \omega_{\alpha_1} + \cdots + \int_M \omega_{\alpha_k} \in \mathbb{R} \tag{6.3}$$

This proves existence $\int_{(M, \mathcal{O}, \mathcal{U}, \{\rho_\alpha\})} : \Omega_c^n(M) \to \mathbb{R}$.

**Uniqueness:** $\forall \omega \in \Omega_c^n(M)$, $\exists \omega_1, \cdots, \omega_k \in \Omega_s^n(M)$, $\omega = \omega_1 + \cdots + \omega_n$ as the claim proved.

So $\displaystyle\int_M \omega = \sum_{i=1}^n \int_M \omega_i$ is uniquely defined. □

**Remark 6.9.** We actually obtain that each $\omega \in \Omega_c^n(M)$ can be expressed as $\sum_{k=1}^{n} \omega_k$ where $\omega_k \in \Omega_s^n(M)$.

**Proposition 6.10.**

1. *$f : (M, \mathcal{O}_M) \xrightarrow{\cong} (N, \mathcal{O}_N)$. If $f$ is orientation preserving, then $\int_M f^*(\omega) = \int_N \omega$. If $f$ is orientation reversing, then $\int_M f^*(M) = -\int_N \omega$.*

2. $\int_{(M,\mathcal{I})} \omega = -\int_{(M,\overline{\mathcal{O}})} \omega$.

3. *If $\mathrm{Supp}(\omega) \subset U \overset{open}{\subset} M$, then $\int_M \omega = \int_U \omega$.*

*Proof.* Leave as exercise. □

## 6.3 Smooth Manifold with Boundary

Now let $M$ be the smooth manifold with boundary, $\partial M = N$. *i.e.* $M$ has a smooth atlas $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \overset{\text{open}}{\subset} \mathbb{R}_{\leqslant 0} \times \mathbb{R}^{n-1}\}$.

We can define $T_pM$, $TM$, $\mathrm{Alt}^k(M)$, $\Omega^k(M)$, $\Omega_c^k(M)$, $\Omega_s^k(M)$ and orientation similar as before.

For $p \in \partial M$, $X \in T_pM$ is called **outward** if $\exists$ local chart $\varphi : U \xrightarrow{\cong} V$ around $p$ *s.t.*

$$\varphi_{p,*}(X) = a_1 \partial x^1 + \cdots + a_n \partial x^n \text{ with } a_1 > 0$$

Recall that if $M$ is $n$ dimensional manifold with boundary, then $N = \partial M$ is a $n-1$ dimensional manifold without boundary.

**Proposition 6.11.** *For any orientation $\mathcal{O}_M$ on $M$, $\exists$ a unique induced orientation $\mathcal{O}_N$ on $N = \partial M$ s.t. $\forall p \in N$, $X = T_pM$ outward, $e_2, \cdots, e_n \in T_pN$ is an oriented basis. Moreover, $(X, e_2, \cdots, e_n)$ is oriented basis of $T_pM$.*

*Proof.* Take oriented atlas $\mathcal{U} = \{\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha\}_{\alpha\in\mathcal{A}}$.

We have $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \{0\} \times \mathbb{R}^{n-1}$.

Define $\psi_\alpha : N \cap U_\alpha \xrightarrow{\cong} (\{0\} \times \mathbb{R}^{n-1}) \cap U_\alpha$. Then $\mathcal{U}'\{\psi_\alpha\}$ is a smooth atlas for $N$.

$\mathcal{U}$ is oriented implies $\mathcal{U}$ is positive, so is $\mathcal{U}$. So there exists the unique $\mathcal{O}_N$ *s.t.* $\mathcal{U}'$ is oriented. □

**Theorem 6.12** (Stokes' Theorem). *$M$ is $n$ dimensional manifold with boundary, oriented by $\mathcal{O}_M$. $N = \partial M$, with induced orientation $\mathcal{O}_N$. $\iota : N \hookrightarrow M$ is the inclusion map. Then $\forall \omega \in \Omega_c^{n-1}(M)$, we have*

$$\int_M \mathrm{d}\omega = \int_N \iota^*(\omega) \tag{6.4}$$

*Proof.* By Remark 6.9 $\forall\omega \in \Omega_c^{n-1}(M)$, $\omega = \omega_1 + \cdots + \omega_k$, $\omega_j \in \Omega_s^n(M)$.

By linearity, we may assume $\omega \in \Omega_s^n(M)$, $\mathrm{Supp}(\omega) \subset U_\alpha$ for chart $\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha$.

$\varphi_\alpha^{-1,*}(\omega) \in \Omega_c^n(V_\alpha)$ induces $\omega' \in \Omega_c^n(\mathbb{R}_{\leqslant 0} \times \mathbb{R}^{n-1})$ if we extend it by 0.

By considering $\omega'$ instead of $\omega$, we may assume $M = \mathbb{R}_{\leqslant 0} \times \mathbb{R}^{n-1}$ and we just need to prove

$$\int_{\mathbb{R}^{n-1}} \iota^*(\omega') = \int_{\mathbb{R}_{\leqslant 0}\times\mathbb{R}^{n-1}} \mathrm{d}\omega'$$

Let $\omega' = \sum_{k=1}^n f^k \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^k} \wedge \cdots \wedge \mathrm{d}x^n$.

By linearity, we may assume $\omega' = f^k \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^k} \wedge \cdots \wedge \mathrm{d}x^n$.

$$\begin{aligned} \mathrm{d}\omega' &= \left(\sum_{i=1}^n \frac{\partial f^k}{\partial x^i}\mathrm{d}x^i\right) \\ &= \frac{\partial f^k}{\partial x^k}(-1)^{k-1}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \end{aligned} \tag{6.5}$$

For $k = 1$,

$$
\begin{aligned}
\int_{\mathbb{R}_{\leqslant 0}\times\mathbb{R}^{n-1}} \mathrm{d}\omega' &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\leqslant 0}} \frac{\partial f^1}{\partial x^1}\mathrm{d}x^1\right) \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \int_{\mathbb{R}^{n-1}} \left(f^1(0,x_2,\cdots,x_n) - \lim_{x_1\to\infty} f^1(x_1,x_2,\cdots,x_n)\right) \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \\
&= \int_{\mathbb{R}^{n-1}} f^1(0,x^2,\cdots,x^n)\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \int_{\mathbb{R}^{n-1}} \iota^*(\omega')
\end{aligned}
$$

For $k \neq 1$,

$$
\begin{aligned}
\int_{\mathbb{R}_{\leqslant 0}\times\mathbb{R}^{n-1}} \mathrm{d}\omega' &= \int_{\mathbb{R}_{\leqslant 0}\times\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}_{\leqslant 0}} \frac{\partial f^k}{\partial x^k}\mathrm{d}x^k\right) \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{x_k\to+\infty} f^k(x_1,x_2,\cdots,0,\cdots,x_n) - \lim_{x_k\to-\infty} f^k(x_1,x_2,\cdots,x_n)\right) \\
&\quad \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \widehat{\mathrm{d}x_k} \wedge \cdots \wedge \mathrm{d}x^n \\
&= 0 \\
&= \int_{\partial M} \iota^*(\omega')
\end{aligned}
$$

□

**Theorem 6.13.** *For any smooth orientable compact manifold $M$, $\partial M$ is not a **retract** of $M$, i.e. there is no continuous map $r : M \to \partial M$ s.t. $r|_{\partial M} = \mathrm{id}_{\partial M}$*

*Proof.* Assume such retraction $r$ exists. After homotopy, we may assume $r$ is smooth followed form Whitney approximation Theorem 1.41

Pick an orientation on $M$ and such an orientation induces orientation on $\partial M$ and hence, there exists an orientation form $\omega \in \Omega^{n-1}(\partial M)$ *s.t.* $\omega$ is nowhere vanishing.

Then $\int_{\partial M} \omega \neq 0$.

Noticed that $r \circ i = \mathrm{id}$, we have $\int_{\partial M} \omega = \int_{\partial M} \iota^*(r^*(\omega)) = \int_M \mathrm{d}(r^*(\omega)) = \int_M r^*(\mathrm{d}\omega) = 0$ □

**Corollary 6.14.** *$f : D^n \to D^n$ continuous. Then $\exists x \in D^n$ s.t. $f(x) = x$.*

*Proof.* Suppose there is no $x$ *s.t.* $f(x) = x$. Let $l_x$ be the unique ray form $f(x)$ to $x$.

Define $r : D^n \to S^{n-1}$ by $r(x) = l_x \cap S^{n-1}$. Then $r$ is continuous, $r(x) = x$ for $x \in S^{n-1}$, which is contradiction to the previous theorem. □

## 6.4 Riemannian Metric

A **Riemannian metric** $g$ on $M$ is a smooth section of $T^*M \otimes T^*M$ *s.t.* $\forall p \in M$, $g_p : T_pM \otimes T_pM \to \mathbb{R}$ is symmetric and positive definite.

Or equivalently, $g_p$ is a bilinear map from $T_pM \times T_pM \to \mathbb{R}$ *s.t.*

$$g_p(v, w) = g_p(w, v), \quad g_p(v, v) \geqslant 0 \text{ with equality holds iff } v = 0$$

We also write $g_p(v, w)$ as $\langle v, w \rangle_p$ $v, w \in T_pM$.

In a local chart $(U, x^1, \cdots, x^n)$,

$$g|_U = \sum_{1 \leqslant i,j \leqslant n} g_{i,j} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where $g_{i,j} \in C^\infty(U, \mathbb{R})$, $g_{i,j}(p) = \left\langle \frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p \right\rangle_p$.

We will abbreviate Riemannian metric to just metric in this lecture, just for convenience.

**Example 6.15.**

(1) Euclidean metric on $\mathbb{R}^n$, $g_{std} = \sum_{i=1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^j$.

(2) Given any $N \overset{f}{\looparrowright} M$, any metric $g_M$ on $M$, we can pullback $g_M$ to metric on $g_N$ by $\langle V, W\rangle_p := \left\langle f_{p,*(V)}, f_{p,*}(W)\right\rangle_{f(p)} \in \mathbb{R}, \forall p \in N, V, W \in T_pN$.

When $f$ is an embedding, also write $f^*(g_M)$ as $g_M|_N$

(3) For $S^{n-1} \hookrightarrow \mathbb{R}^n$, $g_{std}|_{S^{n-1}} = g_{S^{n-1}}$, $T_pS^{n-1} \cong p^\perp$. So $\langle v, w\rangle_p = \langle v, w\rangle_{\mathbb{R}^n}$. Hence $S^{n-1}$ has **constant positive sectional curvature**

(4) For $M = \overset{\circ}{D}{}^n$, $g = \dfrac{\sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i}{(1 - \sum_{i=1}^n x^i)^2}$ called **Poincaré disk model**, has **constant sectional curvature -1**.

**Proposition 6.16.** *Any smooth manifold has a Riemannian metric,*

*Proof.* Take a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$, take partition of unity $\rho_\alpha : M \to [0,1]$ subordinate to $\mathcal{U}$.

On each $(U_\alpha, x^1, \cdots, x^n)$, take the standard metric

$$g_\alpha = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$$

Then $g = \sum_\alpha \rho_\alpha g_\alpha \in \Gamma(T^*M \otimes T^*M)$.

For $\forall p \in \alpha$, there exists finitely many $\alpha_i$ *s.t.* $g_p = \sum_{i=1}^m \rho_{\alpha_i}(p) g_{\alpha_i}$. Positive linear combination of positive definite symmetric form is still positive definite and symmetric. Thus, $g$ is what we need. □

Consider $\mathbb{R}^n \hookrightarrow E \to M$ a smooth vector bundle.

A **Riemannian metric** on $E$ is a smooth section $g \in \Gamma(E^* \otimes E^*)$ *s.t.* $\forall p \in M$, $g_p$ is a symmetric, positive definite bilinear form.

**Proposition 6.17.** *Any smooth vector bundle has a metric.*

*Proof.* It suffices to replace $T_pM$ to vector bundle in the previous proof. $\square$

**Corollary 6.18.** *For any smooth vector bundle $E$, $E$ is isomorphic to $E^*$. i.e. $\exists \rho$ diffeomorphism s.t. $\rho$ restricts to linear isomorphism $E_p \to E_p^*M$ for any $p \in M$.*

*Proof.* Pick a metric $g$ on $E$. Define $E \xrightarrow{\rho} E^*, v \mapsto g_p(v, -)$. It is easy to check $\rho$ is an isomorphism. $\square$

In particular, $TM \cong T^*M$.

**Theorem 6.19.** *Let $M$ be oriented manifold. Then any Riemannian metric $g$ determines an oriented $n$-form $\mathrm{Vol} \in \Omega^n(M)$, called **volume form**.*

*Proof.* Take oriented chart $(U_\alpha, x^1, \cdots, x^n)$, we get vector fields $\partial x^i$ on $U_\alpha$.

Apply Gram-Schmidt process,

$$e_1 = \frac{\partial x^1}{||\partial x^1||_g}, e_2 = \frac{\partial x^2 - \langle \partial x^2, e_1 \rangle_g}{||\partial x^2 - \langle \partial x^2, e_1 \rangle_g||}, e_3 = \cdots$$

We get smooth orthonormal vector fields $e_1, \cdots, e_n$ on $U_\alpha$. Let $e_1^*, \cdots, e_n^* \in \Omega^1(U_\alpha)$ be the dual of $\{e_i\}$.

**Set** $\omega_\alpha := e_1^* \wedge \cdots \wedge e_n^* \in \Omega^n(U_\alpha)$.

Now suppose we have another chart $(U_\beta, y^1, \cdots, y^n) \rightsquigarrow e_\beta = e_1'^* \wedge \cdots \wedge e_n'^*$.

Then $\forall p \in U_\alpha \cap U_\beta$, $\{e_{i,p}\}, \{e'_{i,p}\}$ are both oriented orthogonormal basis of $(T_pM, g_p)$.

Then there exists $A \in SO(n)$ *s.t.*

$$\begin{pmatrix} e_{1,p} \\ \vdots \\ e_{n,p} \end{pmatrix} = A \begin{pmatrix} e'_{1,p} \\ \vdots \\ e'_{n,p} \end{pmatrix}$$

Then

$$\begin{pmatrix} e_{1,p}^* \\ \vdots \\ e_{n,p}^* \end{pmatrix} = A^T \begin{pmatrix} e_{1,p}'^* \\ \vdots \\ e_{n,p}'^* \end{pmatrix}$$

So $e_{1,p}'^* \wedge \cdots \wedge e_{n,p}'^* = \det(A^T) e_{1,p}^* \wedge \cdots \wedge e_{n,p}^* \Rightarrow \omega_{\alpha,p} = \omega_{\beta,p}$.

Therefore, $\omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha \cap U_\beta}$. So $\{\omega_\alpha\}$ induces a unique $\mathrm{Vol} \in \Omega^n(M)$ *s.t.* $\mathrm{Vol}|_{U_\alpha} = \omega_\alpha$ □

**Remark 6.20.** In this proof, we know that the wedge product of orthogonormal basis at any point are all the same except signs.

**Proposition 6.21** (Calculation of volume form in local chart)**.** *In a local chart* $(U, x^1, \cdots, x^n)$, $g|_U = \sum_{1 \leqslant i,j \leqslant n} g_{i,j} \mathrm{d}x^i \otimes \mathrm{d}x^j$, *then*

$$\mathrm{Vol} = \sqrt{\det(g_{ij})} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{6.6}$$

*Proof.* If $\begin{pmatrix} \partial x^1 \\ \vdots \\ \partial x^n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$, where $A : U \to \mathrm{GL}_n(\mathbb{R})$.

Then $\begin{pmatrix} e_1^* \\ \vdots \\ e_n^* \end{pmatrix} = A^T \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^n \end{pmatrix} \Rightarrow$

$$\mathrm{Vol} = e_1^* \wedge \cdots \wedge e_n^* = \det(A^T) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \sqrt{\det(g_{ij})} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

The final equality is because

$$(g_{ij}) = \begin{pmatrix} \partial x^1 \\ \vdots \\ \partial x^n \end{pmatrix}_g \begin{pmatrix} \partial x^1 & \cdots & \partial x^n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}_g \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} A^T = AA^T$$

The multiplication of matrix is defined by the Riemannian metric $\langle -, - \rangle_{g_p}$ □

**Example 6.22.** For $\omega_0 \in \Omega^{n-1}(\mathbb{R}^n)$ defined by

$$\omega_{0,x}(v_1, \cdots, v_{n-1}) = \det(x, v_1, \cdots, v_{n-1})$$

Then $\omega_{0,x}(\partial x^1, \cdots, \widehat{\partial x^k}, \cdots, \partial x^n) = (-1)^{i-1}x^i$. So

$$\omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

And we have $\omega_0|_{S^{n-1}} = \mathrm{Vol}_{S^{n-1}}$.

*Proof.* For $x \in S^{n-1}$, pick an orthogonormal basis $(e_2, \cdots, e_n)$ for $T_x S^{n-1} \cong x^\perp \subset \mathbb{R}^n$. Then $(x, e_2, \cdots, e_n)$ is an oriented orthonormal basis for $\mathbb{R}^n$

Then $\omega_{0,x}(e_2, \cdots, e_n) = \det(x, e_2, \cdots, e_n) = 1$. So $(\omega_0|_{S^{n-1}})_x = e_2^* \wedge \cdots \wedge e_n^* = (\mathrm{Vol}_{S^{n-1}})_x$. So $\omega_0|_{S^{n-1}} = \mathrm{Vol}_{S^{n-1}}$ □

**Remark 6.23.** In this example, we see that it is usual to view the differential $n$-form as the anti-symmetric map $T^n M \to \mathbb{R}$ and it is uniquely determined by the image of the orthogonormal basis of tangent space.

$\omega_0$ is not closed since $\mathrm{d}\omega_0 = n \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Actually, there is no closed $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ *s.t.* $\omega|_{S^{n-1}} = \mathrm{Vol}_{S^{n-1}}$, since $\mathrm{Vol}(S^{n-1}) > 0$ but $\int_{S^{n-1}} \omega|_{S^{n-1}} = \int_{D^n} \mathrm{d}\omega = 0$.

However, there exists $\omega_1 \in \Omega^{n-1}(\mathbb{R}^n \backslash \{0\})$ *s.t.* $\mathrm{d}\omega_1 = 0$. $\omega_1|_{S^{n-1}} = \mathrm{Vol}_{S^{n-1}}$.

Indeed, for $\gamma : \mathbb{R}^n\backslash\{0\} \to S^{n-1}$, $x \mapsto \frac{x}{|x|}$, let $\omega_1 = \gamma^*(\mathrm{Vol}_{S^{n-1}})$. Then $\mathrm{d}\omega_1 = 0$, $\omega_1|_{S^{n-1}} = \mathrm{Vol}_{S^{n-1}}$.

**Exercise 6.24.**

$$\omega_1 = \sum_{i=1}^{n}(-1)^{i-1}\frac{1}{|x|^n}x_i\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x^n$$

## 6.5 Orientability of $\mathbb{RP}^{n-1}$

Define the $A : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto -x$, $A|_{S^{n-1}} : S^{n-1} \to S^{n-1}$. Then

$$\begin{aligned}(A^*(\omega_0))_x (V_2, \cdots, V_n) &= \omega_{0,-x}(-V_2, \cdots, -V_n)\\ &= \det(-x, -V_2, \cdots, V_n)\\ &= (-1)^n \det(x, V_2, \cdots, V_n)\\ &= (-1)^n\omega_{0,x}(V_2, \cdots, V_n)\end{aligned} \tag{6.7}$$

Therefore, $A^*\omega_0 = (-1)^n\omega_0 \Rightarrow (A|_{S^{n-1}})^* (\mathrm{Vol}_{S^{n-1}}) = (-1)^n\mathrm{Vol}_{S^{n-1}}$.

① If $n$ is even, $(A|_{S^{n-1}})^* (\mathrm{Vol}_{S^{n-1}}) = \mathrm{Vol}_{S^{n-1}}$.

Define $\mathbb{RP}^{n-1} = S^{n-1}/_{x\sim Ax}$. Then $\mathrm{Vol}_{S^{n-1}}$ induces a nowhere vanishing form $\omega \in \Omega^{n-1}(\mathbb{RP}^{n-1})$ if $n$ is even. So $\mathbb{RP}^{n-1}$ is orientable, $\omega$ is the volume form $\mathrm{Vol}_{\mathbb{RP}^{n-1}}$

**Proposition 6.25.** *We have the identification:*

$$\mathbb{RP}^1 = S^1, \mathbb{RP}^3 \cong \mathrm{SO}(3) \cong \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | u\perp v, |u| = |v| = 1\}$$

② If $n$ is odd, $(A|_{S^{n-1}})^*(\mathrm{Vol}|_{S^{n-1}}) = -\mathrm{Vol}|_{S^{n-1}}$. We claim that

**Proposition 6.26.** $\mathbb{RP}^{n-1}$ *is unorientable for $n$ odd.*

*Proof.* Suppose $\mathbb{RP}^{n-1}$ is orientable. Then $\exists$ nowhere vanishing orientation form

$\omega' \in \Omega^{n-1}(\mathbb{RP}^{n-1})$. Define

$$q : S^{n-1} \to \mathbb{RP}^{n-1}$$
$$x \mapsto [x]$$

Then $q^*(\omega') = f\mathrm{Vol}_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$ for $f \in C^\infty(S^{n-1})$ nowhere vanishing.

By $q \circ A = q$, $A^*(q^*(\omega')) = q^*(\omega') \Rightarrow A^*(f\mathrm{Vol}_{S^{n-1}}) = f\mathrm{Vol}_{S^{n-1}}$. Then

$$(f \circ A) \cdot A^*(\mathrm{Vol}_{S^{n-1}}) = f\mathrm{Vol}_{S^{n-1}} \Rightarrow f \circ A = -f \tag{6.8}$$

So $f(-x) = -f(x)$, $\forall x \in S^{n-1}$. $\Rightarrow f$ can't be nowhere vanshing, which causes contradiction! □

**Theorem 6.27.** $\mathbb{RP}^n$ *is orientable iff* $n$ *is odd.*

## 6.6 Tensor Field and Lie Derivative

For $M$ smooth manifold, $a, n \in \mathbb{N}$.

A $(a,b)$-**tensor field** is a smooth section $\tau$ of $\left(\bigotimes_a TM\right) \bigotimes \left(\bigotimes_b T^*M\right)$.

A $(a,b)$-tensor field is called **covariant tensor field** if $a = 0$.

A $(a,b)$-tensor field is called **contravariant tensor field** if $b = 0$.

A $(a,b)$-tensor field is called **mixed tensor field** if $a \neq 0, b \neq 0$.

Under local chart $(U, x^1, \cdots, x^n)$, a $(a,b)$-tensor field $\tau$ can be written as

$$\tau|_U = \sum_{\substack{1 \leqslant j_1 < \cdots < j_b \leqslant n \\ 1 \leqslant i_1 < \cdots < i_a \leqslant n}} \tau^{i_1, \cdots, i_a}_{j_1, \cdots, j_b}\, \partial x^{i_1} \otimes \cdots \otimes \partial x^{i_a} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_b} \tag{6.9}$$

**Example 6.28.**

(1) $f \in C^\infty(M)$ is a $(0,0)$-tensor field.

(2) Vector field is a $(1,0)$-tensor field.

(3) $k$-form is an anti-symmetric $(0,k)$-tensor field.

(4) A Riemannian metric is a symmetric $(0,2)$-tensor field.

Given vector field $X$, $(a,b)$-tensor field $\tau$, one can define the **Lie derivative** $\mathcal{L}_X\tau$ be a $(a,b)$-tensor field.

We focus on the case $a = 0$, *i.e.* covariant vector field. (Indeed, we have defined the Lie derivative for the contravariant field when $b = 0$)

For $p \in M$, $\{\varphi_t : U \to M\}_{r\in(-\varepsilon,\varepsilon)}$ local flow for $X$. Define $\mathcal{L}_X\tau$ as follows:

For $X \in \Gamma(TM)$, $\varphi_{t,*} : T_pM \to T_{\varphi_t(q)}M$ induces $\varphi_t^* : T^*_{\varphi_t(p)}M \to T^*_pM$ and

$$\varphi_t^* : \bigotimes_b T^*_{\varphi_t(p)}M \to \bigotimes_b T^*_pM$$

Since $\tau_{\varphi_t(p)} \in \bigotimes_b T^*_{\varphi_t(p)}M$, $\tau_p \in \bigotimes_b T^*_pM$, define

$$\left(\mathcal{L}_X\tau\right)_p = \lim_{t\to 0}\frac{\varphi_t^*(\tau_{\varphi_t(p)}) - \tau_p}{t} \tag{6.10}$$

Equivalently, $\mathcal{L}_X\tau|_U = \lim\limits_{t\to 0}\dfrac{\varphi_t^*(\tau)|_U - \tau|_U}{t}$.

**Lemma 6.29.** *$X, \tau$ smooth $\Rightarrow \mathcal{L}_X\tau$ smooth.*

*Proof.* See Lee's Book □

$\mathcal{L}_X\tau$ describes the change of $\tau$ under the (local) flow generated by $X$.

For simplicity, let's assume $X$ is complete. Let $\varphi_t : M \to M$ be the global flow for $X$. We say a covariant tensor field $\tau \in \Gamma\left(\bigotimes_b T^*M\right)$ is invariant under $\varphi_t$ or invariant under $X$ if $\varphi_t^*(\tau) = \tau$.

**Proposition 6.30.**

- *$\tau$ is invariant under $X$ if and only if $\mathcal{L}_X \tau = 0$*
- $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*(\tau)|_{t=0} = \mathcal{L}_X \tau$

*Proof.* The second part and the necessary of the first part is direct from the definition.

We only prove the sufficiency for the first part. Assume $\mathcal{L}_X \tau = 0$.

Consider the map $\mathbb{R} \xrightarrow{f} \bigotimes_b T_p^* M$, $t \mapsto \varphi_t^*(\tau_{\varphi_t(p)})$.

Then $\forall t_0 \in \mathbb{R}$, $f(t) = \varphi_{t_0}^*(\varphi_{t-t_0}^*(\tau_{\varphi_t(p)}))$. So

$$
\begin{aligned}
\frac{\mathrm{d}f(t)}{\mathrm{d}t}|_{t=t_0} &= \lim_{t\to t_0} \frac{\varphi_{t_0}^*(\varphi_{t-t_0}^*(\tau_{\varphi_t(p)})) - \varphi_{t_0}^*(\tau_{\varphi_{t_0}(p)})}{t-t_0} \\
&\xlongequal{s=t-t_0} \varphi_{t_0}^* \left( \lim_{s\to 0} \frac{\varphi_s^*(\tau_{\varphi_s(\varphi_{t_0}(p))}) - \tau_{\varphi_{t_0}(p)}}{s} \right) \\
&= \varphi_{t_0}^* \left(\mathcal{L}_X \tau\right)_{\varphi_{t_0}(p)} \\
&= 0
\end{aligned}
\tag{6.11}
$$

□

**Definition 6.31.** Given Riemannian manifold $(M, g)$, a vector field $X$ is called a **Killing vector field** if $\mathcal{L}_X g = 0$. Or equivalently, the flow generated by $X$ is an isometry.

**Example 6.32.**

$$
\begin{aligned}
\left\{\text{Killing vector field on } (S^2, g_{S^2})\right\} &\cong \mathfrak{so}(3) \\
&= \mathrm{Span}\left(x\partial y - y\partial x, y\partial z - z\partial y, z\partial x - x\partial z\right)
\end{aligned}
$$

**Lemma 6.33.**

*(1)* $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$

*(2)* $\mathcal{L}_X(\mathrm{d}\omega) = \mathrm{d}(\mathcal{L}_X \omega)$

*Proof.* (1)

$$
\begin{aligned}
(\mathcal{L}_X(\omega \wedge \eta))_p &= \lim_{t\to 0} \frac{\varphi_t^*(\omega \wedge \eta)_{\varphi_t(p)} - (\omega \wedge \eta)_p}{t} \\
&= \lim_{t\to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) \wedge \varphi_t^*(\eta_{\varphi_t(p)}) - \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \eta_p + \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \eta_p - \omega_p \wedge \eta_p}{t} \\
&= \lim_{t\to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} \wedge \eta_p + \lim_{t\to 0} \varphi_t^*(\omega_{\varphi_t(p)}) \wedge \lim_{t\to 0} \frac{\varphi_t^*(\eta_{\varphi_t(p)}) - \eta_p}{t} \\
&= (\mathcal{L}_X\omega)_p \wedge \eta_p + \omega_p \wedge (\mathcal{L}_X\eta)_p
\end{aligned}
\tag{6.12}
$$

(2)

$$
\begin{aligned}
\mathcal{L}_X \mathrm{d}\omega &= \lim_{t\to 0} \frac{\varphi_t^*(\mathrm{d}\omega) - \mathrm{d}(\omega)}{t} \\
&= \lim_{t\to 0} \frac{\mathrm{d}(\varphi_t^*(\omega) - \omega)}{t} \\
&= \mathrm{d}\left(\lim_{t\to 0} \frac{\varphi_t^*(\omega) - \omega}{t}\right) \\
&= \mathrm{d}\mathcal{L}_X(\omega)
\end{aligned}
$$

□

Given $X \in \Gamma(TM)$, define **contraction** $X \lrcorner - : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$(X \lrcorner \alpha)(Y_2, \cdots, Y_k) := \alpha(X, Y_2, \cdots, Y_k) \text{ for } Y_2, \cdots, Y_k \in \Gamma(TM)$$

Often abbreviated to $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$.

**Example 6.34.** $X = f\partial x^i$, $\alpha = g\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$. Then

$$
x \lrcorner \alpha = \begin{cases} 0, & i \notin \{i_1, \cdots, i_n\} \\ (-1)^{k-1} fg\mathrm{d}x^{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x^{i_k}} \wedge \cdots \wedge \mathrm{d}x^{i_n}, & i = i_k \end{cases}
\tag{6.13}
$$

**Theorem 6.35** (Cartan's Magical Formula). $\mathcal{L}_X(\omega) = X \lrcorner \mathrm{d}\omega + \mathrm{d}(X \lrcorner \omega)$. *Also written as*

$$\mathcal{L}_X(\omega) = \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X \omega)$$

*Proof.* Need to show $(\mathcal{L}_X\omega)_p = (X \lrcorner (\mathrm{d}\omega))_p + (\mathrm{d}(X \lrcorner \omega))_p$.

For $X_p \neq 0$. By canonical form theorem 4.1, there exists local chart around $p$ s.t. $X|_U = \partial x^1$.

Let $\omega = \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant n} f^{i_1, \cdots, i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$. By linearity we may assume $\omega = f \mathrm{d}x^{i_1} \wedge \cdots \wedge x^{i_k}$, $i_1 < \cdots < i_k$.

For $i_1 = 1$,

$$\varphi_t(x^1, \cdots, x^n) = (x^1 + t, \cdots, x^n)$$

$$\begin{aligned} X \lrcorner \mathrm{d}\omega &= \frac{\partial}{\partial x^1} \lrcorner \left( \sum_{1 \leqslant j \leqslant n} \frac{\partial f}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \right) \\ &\overset{(6.13)}{=} - \sum_{j \notin \{i_1, \cdots, i_k\}} \frac{\partial f}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \end{aligned}$$

$$(X \lrcorner \omega) \overset{(6.13)}{=} f \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Then

$$\mathrm{d}(X \lrcorner \omega) = \sum_{j \notin \{i_2, \cdots, i_k\}} \frac{\partial f}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

So

$$(\mathrm{d}(X \lrcorner \omega) + X \lrcorner \omega)_p = \frac{\partial f}{\partial x^1} \mathrm{d}x^1 \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge x^{i_k} = (\mathcal{L}_X \omega)_p$$

For the case $i_1 \neq 1$,

$$X \lrcorner \mathrm{d}\omega = \frac{\partial}{\partial x^1} \lrcorner \left( \sum_{1 \leqslant j \leqslant n} \frac{\partial f}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \right)$$
$$\overset{(6.13)}{=} \frac{\partial f}{\partial x^1} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

$$(X \lrcorner \omega) \overset{(6.13)}{=} 0$$

Thus,

$$(\mathrm{d}(X \lrcorner \omega) + X \lrcorner \omega)_p = \frac{\partial f}{\partial x^1} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = (\mathcal{L}_X \omega)_p$$

Now we consider the case that $X_p = 0$. For $p \in \mathrm{Supp}(X)$, it is true using continuity. If $p \notin \mathrm{Supp}(X)$, then $\exists$ neighborhood $U$ of $p$ *s.t.* $X|_U \equiv 0$. So the two sides of the equation equals 0. □

### 6.6.1 Divergence of vector fields

**Definition 6.36.** For Riemannian metric space $(M, g)$, $X \in \Gamma(TM)$. Define $\mathrm{div}X \in C^\infty(M)$ by

$$\mathrm{d}(\iota_X \mathrm{Vol}) = \mathrm{div}(X)\mathrm{Vol} \tag{6.14}$$

Then by Cartan's formula,

$$\mathcal{L}_X \mathrm{Vol} = \mathrm{d}(\iota_X \mathrm{Vol}) = \mathrm{div}(X)\mathrm{Vol} \tag{6.15}$$

Let $\varphi_t : U \to M$ be the flow for $X$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^*(\mathrm{Vol})|_{t=t_0} = \varphi_{t_0}^*(\mathcal{L}_X \mathrm{Vol}) = \varphi_{t_0}^*(\mathrm{div}X \cdot \mathrm{Vol}) \tag{6.16}$$

If $D$ is an integration domain of $M$, a small ball for example. Then define

$$\mathrm{Vol}(D) = \int_D \mathrm{Vol} \tag{6.17}$$

Now $\mathrm{Vol}(\varphi_t(D)) = \int_{\varphi_t(D)} \mathrm{Vol} = \int_D \varphi_t^*(\mathrm{Vol})$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(\varphi_t(D)) &= \int_D \varphi_t^*(\mathrm{div}X\mathrm{Vol}) \\ &= \int_{\varphi_t(D)} \mathrm{div}X\mathrm{Vol} \end{aligned} \tag{6.18}$$

If $\mathrm{div}X|_p > 0$ for all $p$, then $\varphi_t$ is volume increasing. If $\mathrm{div}X|_p = 0$ for all $p$, then $\varphi_t$ is volume preserving. If $\mathrm{div}X|_p < 0$ for all $p$, then $\varphi_t$ is volume decreasing.

### 6.6.2 Hamiltonian vector fields on symplectic manifolds

A **symplectic structure** on $M$ is a 2-form $\omega \in \Omega^2(M)$, called **symplectic form** *s.t.* (1) $\mathrm{d}\omega = 0$ (2) $\omega$ is non-degenerate everywhere. *i.e.* For $\forall v \in T_pM \neq 0$, there exists $w \in T_pM$ such that $\omega(v, w) \neq 0$, $\omega^{\frac{\dim M}{2}}$ nowhere vanishing equivalently.

**Example 6.37.** For $M = T^*N$, we have a canonical form $\alpha \in \Omega^1(M)$ defined as follows:

$\forall p \in T_q^*N \subset M$, $v \in T_pM$, it induces a canonical map $\pi : M \to N, p \mapsto q$. Then its pullback $\pi_* : T_pM \to T_qN, v \mapsto \pi_*v$ induces $\alpha_p(v) = p(\pi_*(v)) \in \mathbb{R}$.

Then $\omega = \mathrm{d}\alpha$ is a canonical symplectic structure on $T^*N$.

*Proof.* Take local chart $(U, x^1, \cdots, x^n)$ of $N$,

$$U \xrightarrow{\quad x^i \quad} \widetilde{U} \overset{\text{open}}{\subset} \mathbb{R}^n$$

$$M \supset T^*U \xrightarrow{\cong} T^*\widetilde{U} \cong \widetilde{U} \times \mathbb{R}^n \overset{\text{open}}{\subset} \mathbb{R}^{2n}$$

$$\sum_{i=1}^{n} p_i \mathrm{d}x^i \longmapsto (x^1(q), \cdots, x^n(q), p_1, \cdots, p_n)$$

Then $\alpha = \sum_{i=1}^{n} p_i \mathrm{d}x^i$, $\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}x^i$.

$$\omega^n = n! \mathrm{d}p_1 \wedge \mathrm{d}x_1 \wedge \mathrm{d}p_2 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}x_n$$

So $\omega$ is non-degenerate everywhere. □

**Definition 6.38.** For $(M, \omega)$ symplectic, $f \in C^\infty(M)$, $\mathrm{d}f \in \Omega^1(M)$. $\omega$ is non-degenerate everywhere. Define

$$\iota_{-}\omega : \Gamma(TM) \to \Gamma(T^*M),\ X \mapsto \iota_X \omega \tag{6.19}$$

Define the **Hamiltonian vector field** $X_f$ by $\iota_{X_f}\omega = \mathrm{d}f$. The flow generated by $X_f$ is called the **Hamiltonian flow** $\phi : U \to M$.

By Cartan's formula

$$\mathcal{L}_{X_f}\omega = \iota_{X_f}\mathrm{d}\omega + \mathrm{d}(\iota_{X_f}\omega) = \mathrm{d}(\mathrm{d}f) = 0 \tag{6.20}$$

So the symplectic form is preserved under the Hamiltonian flow.

The motivation to study Hamiltonian flow is that in classical mechanics, $N = \mathbb{R}^n$, $M = T^*N = \mathbb{R}^n \times \mathbb{R}^n$ configuration space. The movement of a system of $m$

partial is given by $\gamma : \mathbb{R} \to M$, $\gamma$ satisfies the Hamiltonian equation

$$\gamma'(t) = X_H|_{\gamma(t)}$$

where $H : M \to \mathbb{R}$ is the Hamiltonian. *i.e.* $\gamma$ is an integral curve of the Hamiltonian flow.

## 6.7 Frobenius Theorem

The motivation is to solve the PDE equation for $f : U \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = \alpha(x, y, f(x, y)) \\ \dfrac{\partial f}{\partial y} = \beta(x, y, f(x, y)) \end{cases} \tag{6.21}$$

where initial value is $f(x_0, y_0) = z_0$.

Consider vector fields on $\mathbb{R}^3$, $X_1 = \dfrac{\partial}{\partial x} + \alpha(x, y, z)\dfrac{\partial}{\partial z}, X_2 = \dfrac{\partial}{\partial y} + \beta(x, y, z)\dfrac{\partial}{\partial z}$. Then $f$ is a solution iff $N : \mathrm{graph}(f) = \{(x, y, f(x, y))\} \hookrightarrow \mathbb{R}^3$ satisfies $(x_0, y_0, z_0) \in N$, $T_pN = \mathrm{Span}(X_{1,p}, X_{2,p})$.

**Definition 6.39.** $M^n$ is a smooth manifold. A $k$-dimensional **tangent distribution** $D$ is a $k$-dimensional linear subspace $D_p \subset T_pM$ at each $p \in M$ *s.t.* $D = \bigsqcup_{p \in M} D_p$ is a smooth subbundle of $TM$.

Equivalently, this means for any $p \in M$, there exists a neighborhood $U$ of $p$, and smooth vector fields $Y_1, \cdots, Y_k$ on $U$ *s.t.* $D_q = \mathrm{Span}\,\langle Y_{1,q}, \cdots, Y_{k,q}\rangle$, $\forall q \in U$.

An immersed submanifold $\varphi : N \looparrowright M$ is called an **integral manifold** for $D$ if $\forall q \in N$, $\varphi_*(T_qN) = D_{\varphi(q)}$. Identify $N$ with $\varphi(N)$ to simplify notation, It can be abbreviated as $T_qN = D_q$,

**Example 6.40.** Every nowhere vanishing vector field is a 1-dimensional distribution, and the integral curve of it is the integral manifold.

**Example 6.41.** For $M = \mathbb{T}^2 = \mathbb{R} \times \mathbb{R}/_{\mathbb{Z}\times\mathbb{Z}}$, $D = \text{Span}(\dfrac{\partial}{\partial x})$. Then the integral submanifold is $S^1 \times \{y\}$. For $D = \text{Span}(\dfrac{\partial}{\partial x} + \sqrt{2} + \dfrac{\partial}{\partial y})$, integral submanifold $\{[y,x] | y - y_0 = \sqrt{2}(x - x_0)\}$ is an immersed but not embedding submanifold dense in $\mathbb{T}^2$.

**Example 6.42.** $M = \mathbb{R}^n$, $D = \text{Span}\left\{\dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^k}\right\}$ has integral manifold $N = \mathbb{R}^k \times \{\mathbf{c}\}$, $\mathbf{c} \in \mathbb{R}^{n-k}$.

**Example 6.43.** $M = \mathbb{R}^n \backslash \{0\}$, $D_p = p^\perp$ is an $n-1$ dimensional distribution with integral submanifold $N = S_r^{n-1} = \{\mathbf{x} : \|\mathbf{x}\| = r\}$.

**Example 6.44.** $M = \mathbb{R}^3$, $D = \text{Span}\left\{\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial y}\right\}$. We claim that $D$ has no integral submanifold.

*Proof of the Claim.* Let $N$ be an integral submanifold, $(0,0,0) \in N$. $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, t \mapsto (t,0,0)$ is an integral curve for $X_1 = \frac{\partial}{\partial x}$. So $(x,0,0) \in N$, for $x \in (-\varepsilon, \varepsilon)$.

Let $\eta : (-\varepsilon', \varepsilon') \to \mathbb{R}^3, t \mapsto (x,t,0)$ be an integral curve for $X_2 = \frac{\partial}{\partial y}$ starting at $(x,0,0)$. So for $(x,y,0) \in N$ for $x \in (-\varepsilon, \varepsilon), y \in (-\varepsilon', \varepsilon')$.

That implies that $N$ contains $X - Y$ plane near $(0,0,0)$ but now $T_pN = \text{Span}\left\{\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right\} \neq D_p$. □

**Remark 6.45.** This example shows that a distribution doesn't have integral manifold always. This property that have integral manifold implies something more than linearity, or in some sense, it should be closed under the action by local flows in that case.

Here we introduce Frobenius theorem to explain it.

**Definition 6.46.** Let $D$ be a $k$-dimensional distribution on $M$.

We say a chart $(U, \varphi)$ on $M$ is **flat** for $D$ if $\varphi(U)$ is a product of connected open sets $U' \times U'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and at points of $U$, $D$ is spanned by the first $k$ coordinate vector fields $\dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^k}$

(1) We say $D$ is **involutive** if for any open $U \subset M$, any smooth sections $X, Y \in \Gamma(D|_U)$, we have $[X, Y] \in \Gamma(D|_U)$.

(2) We say $D$ is **integrable** for any $p \in M$, there exists an integral submanifold $N$ for $D$ *s.t.* $p \in N$

(3) We say $D$ is **completely integrable** if $\forall p \in M$, there exists local chart $(U, x^1, \cdots, x^n)$ around $p$ *s.t.* $D|_U = \text{Span}\left\{\dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^k}\right\}$. *i.e.* There exists a flat chart for $D$ in a neighborhood of every point of $M$.

**Theorem 6.47** (Local Frobenius)**.** *Those definitions above is equivalent.*

*Proof.* Completely integrable $\Rightarrow$ Integrable: In $U$, all submanifolds of the form $\mathbb{R}^k \times \{\mathbf{c}\}$ are integral submanifold for $D$.

Integrable $\Rightarrow$ Involutive: For $X, Y \in \Gamma(D)$, $p \in M$. Let $\iota : N \hookrightarrow M$ be integral submanifold *s.t.* $p \in N$.

Then $[X, Y]|_p = [X|_N, Y|_N]_p \in T_pN = D_p$, since $X, Y$ is $\iota$-related to $X|_N, Y|_N$ resepectively. So $[X, Y] \in \Gamma(D)$.

Involutive $\Rightarrow$ Completely integrable. $\forall p \in M$, take local chart $(V, y^1, \cdots, y^n)$. WLOG, $y(p) = 0$. Then $\pi = (y^1, \cdots, y^k) : V \to \mathbb{R}^k, p \mapsto \mathbf{0}$. It induces $\pi_{q,*} : T_qV \to T_{\pi(q)}\mathbb{R}^k = \mathbb{R}^k$.

Since $D_p$ has dimension $k$, by shrinking $V$, and reordering the coordinate such that $D_q$ is disjoint with $\dfrac{\partial}{\partial x^{k+1}}, \cdots \dfrac{\partial}{\partial x^n}$, we may assume $\pi_{q,*}|_{D_q} : D_q \xrightarrow{\cong} \mathbb{R}^k$, $\forall q \in V$.

Consider $\partial y^i \in \Gamma(T\mathbb{R}^k)$, $1 \leqslant i \leqslant k$. Let $X_i$ be the unique section of $D|_V$ *s.t.* $\pi_*(X_i) = \partial y^i$. $X_i \in \Gamma(D|_V) \subset \Gamma(TM)$. Then $\partial y^i$ is $\pi$-related to $X_i$.

Then $0 = [\partial y^i, \partial y^j]$ is $\pi$-related to $[X_i, X_j]$. So $\pi_*[X_i, X_j]_q = 0, \forall q \in V \Rightarrow [X_i, X_j]_q = 0, \forall q \in V$.

Thus, we obtain linear independent vector fields with $[X_i, X_j] = 0, i \neq j$.

By canonical form of commuting vector fields 4.11, there exists local charts $(U, x^1, \cdots, x^n)$ *s.t.* $X_i = \partial x^i$, $1 \leqslant i \leqslant k$. So $D|_U = \text{Span}\{\partial x^1, \cdots, \partial x^k\}$ □

**Remark 6.48.** Indeed, completely integrable means an embedding locally. So we can find an embedding integrable manifold locally for immersed integrable manifold.

**Lemma 6.49.** *Let $D$ be a $k$-dimensional distribution. Then $D$ is involutive iff there exists an open cover $\mathcal{U}$, $\forall U \in \mathcal{U}$, $\exists X_1, \cdots, X_k \in \Gamma(TU)$ s.t. $D|_U = \text{Span}\{X_1, \cdots, X_k\}$ and $[X_i, X_j] \in \Gamma(D|_U)$*

*Proof.* "$\Rightarrow$" is followed by the definition.

"$\Leftarrow$" Just need to show $D|_U$ is involutive for each $U$.
$\forall X, Y \in \Gamma(D|_U)$, $X = \sum_{i=1}^{n} f^i X_i, Y = \sum_{i=1}^{n} g^i X_i$, $f^i, g^i : U \to \mathbb{R}$. Then

$$[X, Y] = \sum_{i,j} f^i g^j [X_i, X_j] + \sum_{i,j} f^i X_i(g^j) X_j - \sum_{i,j} g^j X_j(f) X_i \in \Gamma(D) \tag{6.22}$$

□

**Corollary 6.50.** $\begin{cases} \dfrac{\partial f}{\partial x} = \alpha(x, y, f(x, y)) \\ \dfrac{\partial f}{\partial y} = \beta(x, y, f(x, y)) \end{cases}$, $f(x_0, y_0) = z_0$ *has local solution for* $\forall (x_0, y_0, z_0)$ *iff*

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}$$

*Proof.* $\Rightarrow$ is because of $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

"$\Leftarrow$" Consider $D = \text{Span}\{\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \beta\frac{\partial}{\partial z}\}$ 2 dimensional distribution on $\mathbb{R}^3$.

$$[\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \beta\frac{\partial}{\partial z}] = \left(\frac{\partial \beta}{\partial x} + \alpha\frac{\partial \beta}{\partial z} - \frac{\partial \alpha}{\partial y} - \beta\frac{\partial \alpha}{\partial z}\right)\frac{\partial}{\partial z} = 0$$

So $D$ is involutive $\Rightarrow \forall(x_0, y_0, z_0), \exists N \looparrowright \mathbb{R}^3 \quad s.t.\ (x_0, y_0, z_0) \in N$, $N$ is an integrable manifold of $D$.

$N \looparrowright \mathbb{R}^3 \xrightarrow{(x,y,z)\mapsto(x,y)} \mathbb{R}^2$ is a submersion. So $N$ can be locally written as $(x, y, f(x, y))$ for $f : U \to \mathbb{R}$ □

A $k$-dimensional **foliation** is a decomposition $M = \bigcup_s N_s \quad s.t.$ (1) Each $N_s$ is an injective immersed $k$-dimensional submanifold. (2) $\forall p \in M$, $\exists$ local chart $(U, x^1, \cdots, x^n) \quad s.t.\ \forall s \in S$, $N_s \cap U = \mathbb{R}^k \times A_s$. $A_s$ is a countable subset of $\mathbb{R}^{n-k}$.

**Example 6.51.** $\mathbb{T}^2 = \bigcup_s N_s$, $N_s = \mathbb{R} \hookrightarrow \mathbb{T}^2$, $t \mapsto (x_0 + t, \sqrt{2}t)$

**Theorem 6.52** (Global Frobenius)**.** *$D$ is an involutive $k$-dimensional distribution $\Rightarrow$ $D$ induces a $k$-dimensional foliation $M = \bigcup_s N_s$ such that each $N_s$ is a maximal integral submanifold of $D$.*

# 7 de-Rham Cohomology

## 7.1 Basic Definition

A **cochain complex** over $\mathbb{Z}$ is a graded abelian group $C = \bigotimes_{b\in\mathbb{Z}} C^n$ with a deg-1 map $\text{d} = \bigotimes_n \left(\text{d}^n : C^n \to C^{n+1}\right) \quad s.t.\ \text{d}^2 = 0$.

d is called **differential** or **boundary map**

The $k$-th **homology**

$$H^k(C, d) := \frac{\ker(\text{d}^k : C^k \to C^{k+1})}{\text{Im}(\text{d}^{k-1} : C^{k-1} \to C^k)} \tag{7.1}$$

The denominator is the set of $k$-**boundary**. The numerator is the set of $k$-**cycles**

**Example 7.1.** $C_{DR}^*(M) = \bigotimes_{n\in\mathbb{Z}} \Omega^n(M)$ is homology equipped with the exterior differential $\mathrm{d}$. And

$$H^k(C_{DR}^*(M), \mathrm{d}) = H_{DR}^* \cong H^k(M; \mathbb{R}) \tag{7.2}$$

where $H^k(M;\mathbb{R})$ is the singular cohomology.

Given cochain complexed $(A, \mathrm{d}_A), (B, \mathrm{d}_B)$, a **chain map** $f : (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$ is a degree 0( *i.e.* $f(A^n) \subset B^n$) group homomorphism *s.t.* $\mathrm{d}_B \circ f = f \circ \mathrm{d}_A$.

If $f$ is a chain map, then $f$ maps $k$-cycles to $k$-cycles and $k$-boundariise to $k$-boundaries. So $f$ induces $f^* : H^k(A, \mathrm{d}_A) \to H^k(B, \mathrm{d}_B)$.

**Example 7.2.** $f :\in C^\infty(M, N) \Rightarrow f^* : \Omega^k(N) \to \Omega^k(M)$ induces a chain map $f^* : C_{DR}^*(N) \to C_{DR}^*(M) \Rightarrow f^* : H_{DR}^*(N) \to H_{DR}^*(M)$ satisfies $(f \circ g)^* = g^* \circ f^*, (\mathrm{id})^* = \mathrm{id}$

Given chain maps $f, g : (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$, a chain homotopy from $f$ to $g$ is a deg-(-1) map $s : A \to B$ *s.t.* $\mathrm{d}_B s + d\mathrm{d}_A = f - g$. In this case we write $f \simeq g$.

$$\begin{array}{ccccc} A^{k-1} & \longrightarrow & A^k & \longrightarrow & A^{k+1} \\ {\scriptstyle g}\downarrow{\scriptstyle f} & \overset{s}{\swarrow} & {\scriptstyle g}\downarrow{\scriptstyle f} & \overset{s}{\swarrow} & {\scriptstyle g}\downarrow{\scriptstyle f} \\ B^{k-1} & \longrightarrow & B^k & \longrightarrow & B^{k+1} \end{array}$$

**Lemma 7.3.** *If* $f \overset{s}{\simeq} g$, *then* $f^* = g^* H^*(A, \mathrm{d}_A) \mapsto H^*(B, \mathrm{d}_B)$

*Proof.* Take $[a] \in H^*(A, \mathrm{d}_A)$, $d_* a = 0$. Then $f(a) - g(a) = \mathrm{d}_B(sa) + s(\mathrm{d}_A a) = \mathrm{d}_B(sa) \in \mathrm{Im}\mathrm{d}_B$. So $f^*[a] = g^*[g]$. □

**Definition 7.4.** Given the map $f : (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$, we say $f$ is a chain homotopy equivalence if $\exists g : (B, \mathrm{d}_B) \to (A, \mathrm{d}_A)$ *s.t.* $g \circ f \simeq \mathrm{id}_A$, $f \circ g \simeq \mathrm{id}_B$. We call such $g$ the **chain homotopy inverse** of $f$. And we say $(A, \mathrm{d}_A), (B, \mathrm{d}_B)$ are **chain homotopic**, denoted as $(A, \mathrm{d}_A) \overset{f}{\simeq} (B, \mathrm{d}_B)$.

**Corollary 7.5.** *If* $(A, \mathrm{d}_A) \overset{f}{\simeq} (B, \mathrm{d}_B)$*, then* $f^* : H^*(A, \mathrm{d}_A) \to H^*(B, \mathrm{d}_B)$ *is an isomorphism.*

*Proof.* Let $g : (B, \mathrm{d}_B) \to (A, \mathrm{d}_A)$ be the chain homotopy inverse of $f$. Then $g^* \circ f^* = \mathrm{id}_{H^*(A,\mathrm{d}_A)}$, $f^* \circ g^* = \mathrm{id}_{H^*(B,\mathrm{d}_B)}$. So $f^*$ is an isomorphism. □

Recall that given smooth manifold $M$ (with boundary), define $C^*_{dR}(M) = \bigotimes_{i \in \mathbb{Z}} \Omega^i(M)$, $d : \Omega^i(M) \to \Omega^{i+1}(M)$ the exterior derivative, $H^*_{dR}(M) = H^*(C_{dR}, \mathrm{d})$.

$f \in C^\infty(M, N) \to$ chain map $f^* : C_{dR}(N) \to C_{dR}(M), a \mapsto f^*(a) \to$ induced map $f^* : H^*_{dR}(N) \to H^*_{dR}(M)$.

**Theorem 7.6.** *Given* $f, g : C^\infty(M, N)$*. Then* $f \simeq g \Rightarrow f^* \simeq g^* : C^*_{dR}(N) \to C^*_{dR}(M)$*. Hence,* $f^* = g^* : H^*_{dR}(N) \to H^*_{dR}(M)$ *by Corollary 7.5*

We will use the Whitney approximation theorem over manifolds.

**Theorem 7.7** (Whitney Approximation Theorem)**.** *Given* $M, N$ *smooth manifolds. Given embedded submanifold* $L \hookrightarrow M$*. Given a continuous map* $f : M \to N$ *s.t.* $f|_L : L \to N$ *is smooth. Then there is a homotopy* $H$ *from* $f$ *to* $g$ *related to* $L$ *s.t.* $g : M \to N$ *is smooth,* $g|_L = f|_L$*.*

## 7.2 Integration along Fibers

For $F$ compact oriented smooth manifold possibly with boundary, $\dim F = l$. $M$ is an $n$-dimensional smooth manifold possibly with boundaries.

Our goal is to define $\int_F - : \Omega^*(F \times M) \to \Omega^{*-l}(M)$ *s.t.* when $l = *$, this map $\int_F : \Omega^l(F) \to \mathbb{R}$ is the usual integration.

Now take $\alpha \in \Omega^k(F \times M)$, we will define $\int_F \alpha \in \Omega^{k-l}(M)$ step by step.

Case i: If $\exists$ chart $(U, x^1, \cdots, x^n)$ for $M$ *s.t.* $\mathrm{Supp}(\alpha) \subset V \times U$ and $\exists$ oriented chart $(V, t^1, \cdots, t^l)$ for $F$. Then

$$\alpha = \sum_{\substack{I\subset\{1,\cdots,l\}\\ J\subset\{1,\cdots,m\}\\ |I|+|J|=k}} \alpha_{IJ}\mathrm{d}t^I \wedge \mathrm{d}x^J,\ \alpha_{IJ} \in C^\infty(V\times U,\mathbb{R}) \tag{7.3}$$

If $I_0 = \{1,\cdots,l\}$, define

$$\begin{aligned} \int_F \alpha_{I_0 J}\mathrm{d}t^{I_0} : & U \to \mathbb{R} \\ & p \mapsto \int_{F\times\{p\}} : \alpha_{I_0 J}|_{F\times\{p\}}\mathrm{d}t^1 \wedge \cdots \mathrm{d}t^l \end{aligned}$$

Now define

$$\int_F \alpha = \sum_{\substack{J\subset\{1,\cdots,m\}\\ |J|=k-l}} \left(\int_F \alpha_{I_0,J}\mathrm{d}t^{I_0}\right)\cdot \mathrm{d}x^I \in \Omega^{k-l}(M) \tag{7.4}$$

It can be generalized to all $\alpha$ and easy the check it is independent with the choice of chart.

Case ii: $\exists$ local chart $(U,x^1,\cdots,x^n)$ for $M$ *s.t.* $\mathrm{Supp}(\alpha)\subset F\times U$. Take an oriented atlas $\mathcal{V} = \{(V_i,\varphi_i)\}_{1\leqslant i\leqslant n}$ for $F$, and partition of unity $\rho_i : F \to [0,1]$ subordinate to $\mathcal{V}$. Let $\widetilde{\rho}_i$ be composition $F\times M \xrightarrow{p} F \xrightarrow{\rho_i} [0,1]$. Then $\mathrm{Supp}(\widetilde{\rho}_{i,\alpha}) \subset V_i \times U$. We define $\int_F \alpha = \sum_{1\leqslant i\leqslant n}\int_F(\widetilde{\rho}_i\alpha)$. Easy to check it is independent with the choice of $U,\mathcal{V},\{\rho_i\}$.

Case iii: For a general $\alpha\in\Omega^k(F\times M)$, take an atlas $\mathcal{U} = \{U_i,\psi_i\}_{i\in I}$ of $M$ and partition of unity $\tau_i : M \to [0,1]$ subordinate to $\mathcal{U}$. Let $\widetilde{\tau}_i$ be composition $F\times M \xrightarrow{p} M \xrightarrow{\tau_i} [0,1]$.

Then $\mathrm{Supp}(\widetilde{\tau}_i\alpha)\subset F\times U_i$. Define $\int_F\alpha = \sum_{i\in I}\int_F(\widetilde{\tau}_i\alpha)$. Easy to check it is independent with the choice of $\mathcal{U},\{\tau_i\}$.

**Remark 7.8.** $\pi : E \to M$ proper submersion. One theorem tells us that $\pi$ is exactly a smooth fiber with compact fiber $F = \pi^{-1}(*)$. For instance, the **Hopf map**

$$\pi : S^3 \to S^2, (z_1, z_2) \in \mathbb{C}^2 \mapsto \frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\}$$

is a smooth fiber with compact fiber.

Then we can generalize above constrution and define $\pi_1 = \int_F - : \Omega^*(E) \to \Omega^{*-\dim(F)}(M)$

**Remark 7.9.** If also works if $F$ is not compact,

$$\int_F - : \Omega^*_{c.F}(F \times M) \to \Omega^*(M) \tag{7.5}$$

where $\Omega^*_{c.F}(F \times M) = \{\alpha \in \Omega^*(F \times M) | \mathrm{Supp}(\alpha) \xrightarrow{\text{Projection}} M \text{ is proper}\}$, viewed as the set of forms with compact fiber.

**Theorem 7.10.** *$F$ is compact, then for any $\alpha \in \Omega^k(F \times M)$, we have*

$$\mathrm{d}\left(\int_F \alpha\right) + \int_F (\mathrm{d}\alpha) = \int_{\partial F} (\alpha|_{\partial F \times M}) \tag{7.6}$$

For the special case $M = *$, $\int_F \mathrm{d}\alpha = \int_{\partial F} \alpha|_{\partial F}$ is exactly the stokes theorem.

For the special case $F = [0, 1]$, $\mathrm{d}\left(\int_F \alpha\right) + \int_F (\mathrm{d}\alpha) = \int_M \alpha|_{\{1\} \times M} - \int_M \alpha|_{\{0\} \times M}$.

We prove only this.

By partition of unity and linearity, we may assume $\mathrm{Supp}(\alpha) \subset [0, 1] \times U$ for some chart $(U, x^1, \cdots, x^m)$ of $M$.

$$\alpha = \sum_{\substack{J \subset \{1, \cdots, m\} \\ \|J| = k}} \alpha_J \mathrm{d}x^J + \sum_{\substack{J \subset \{1, \cdots, m\} \\ \|J| = k-1}} \alpha_J \mathrm{d}t^1 \wedge \mathrm{d}x^J \tag{7.7}$$

By linearity agian, we may assume $\alpha = \alpha_J \mathrm{d}x^J$ or $\alpha = \alpha_J \mathrm{d}t \wedge \mathrm{d}J$.

For $\alpha = \alpha_J \mathrm{d}t \wedge \mathrm{d}J$, we have $\int_F \alpha = 0$, $\mathrm{d}\left(\int_F \alpha\right) = 0$. $\mathrm{d}\alpha = \frac{\partial \alpha_J}{\partial t} \mathrm{d}t \wedge \mathrm{d}\alpha^J +$

$\sum_{1\leqslant i_0\leqslant m}\frac{\partial \alpha_J}{\partial x^{i_0}}\mathrm{d}x^{i_0}\wedge \mathrm{d}x^J$. Then

$$\int_F \mathrm{d}\alpha=\left(\int_0^1\frac{\partial \alpha_J}{\partial t}\mathrm{d}t\right)\cdot \mathrm{d}x^J=\alpha_J|_{\{1\}\times M}\cdot \mathrm{d}x^J-\alpha_J|_{\{0\}\times M}\mathrm{d}x^J=\alpha|_{\{1\}\times M}-\alpha|_{\{0\}\times M} \quad (7.8)$$

So $\mathrm{d}\left(\int_F\alpha\right)+\int_F(\mathrm{d}\alpha)=\int_M\alpha|_{\{1\}\times M}-\int_M\alpha|_{\{0\}\times M}$.

For $\alpha=\alpha_J\mathrm{d}t\wedge \mathrm{d}x^J$, $\int_F\alpha=\left(\int_0^1\alpha_J\mathrm{d}t\right)\mathrm{d}x^J$,

$$\mathrm{d}\int_F\alpha=\sum_{1\leqslant i_0\leqslant m}\frac{\partial}{\partial x^{i_0}}\left(\int_0^1\alpha_J\mathrm{d}t\right)\mathrm{d}x^{i_0}\wedge \mathrm{d}x^J=\sum_{1\leqslant i_0\leqslant m}\left(\int_0^1\frac{\partial \alpha_J}{\partial x^{i_0}}\mathrm{d}t\right)\mathrm{d}x^{i_0}\wedge \mathrm{d}x^J \quad (7.9)$$

$$\mathrm{d}\alpha=-\sum_{1\leqslant i_0\leqslant m}\frac{\partial \alpha_J}{\partial x^{i_0}}\mathrm{d}t\wedge \mathrm{d}x^{i_0}\wedge \mathrm{d}x^J \quad (7.10)$$

So $\int_F\mathrm{d}\alpha=-\sum_{1\leqslant i_0\leqslant m}\left(\int_0^1\frac{\partial \alpha_J}{\partial x^{i_0}}\mathrm{d}t\right)\mathrm{d}x^{i_0}\wedge \mathrm{d}x^J$. Hence $\mathrm{d}\int_F\alpha+\int_F\mathrm{d}\alpha=0=0-0$.

*Proof of Theorem 7.6.* Let $H:[0,1]\times M\to N$ be a homotopy from $f$ to $g$. By Whitney approximation theorem 7.7, we may assume $H$ is a smooth map. Define $s:\Omega^*(N)\to\Omega^{*-1}(M)$, $s(\alpha)=\int_I H^*(\alpha)$. $H^*(\alpha)\in\Omega^*(I\times M)$. Then

$$\begin{aligned}(\mathrm{d}s+s\mathrm{d})\alpha&=\mathrm{d}\int_I H^*\alpha+\int_I H^*(\mathrm{d}\alpha)\\&=\mathrm{d}\int_I H^*\alpha+\int_I \mathrm{d}H^*(\alpha)\\&=H^*\alpha|_{\{1\}\times M}-H^*\alpha|_{\{0\}\times M}\\&=g^*(\alpha)-f^*(\alpha)\end{aligned} \quad (7.11)$$

So $s$ is a chain homotopy between $f^*,g^*:C^*_{dR}(N)\to C^*_{dR}(M)$. □

**Corollary 7.11.** *$f\in C^\infty(M,N)$, $f$ is a homotopy equivalence $\Rightarrow f^*:H^*_{dR}(N)\to H^*_{dR}(M)$ is an isomorphism.*

**Corollary 7.12** (Poincáre Lemma). $H^k_{dR}(\mathbb{R}^n) = \begin{cases} 0 & k > 0 \\ \mathbb{R} & k = 0 \end{cases}$. *i.e. Any closed $k$-form on $\mathbb{R}^n$ with $k > 0$ is exact.*

*Proof.* Take constant $c : \mathbb{R}^n \xrightarrow{\simeq} *$ being a homotopy equivalence

Then $c^* : H^*_{dR}(\text{point}) \to H^k_{dR}(\mathbb{R}^n)$ is an isomorphism. But $\Omega^k(\text{point}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$. So

$$H^k(\mathbb{R}^n) = H^k_{dR}(\text{point}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

□

## 7.3 Mayer-Vietoris sequence

### 7.3.1 Some algebraic constructions

A sequence of maps between groups $\cdots \to G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \to \cdots$ is **exact** at $G_i$ if $\ker(f_i) = \mathrm{Im}(f_{i-1})$. We say the sequence is **exact** if it is exact at every $G_i$.

A **short exact** is an exact sequence of the form $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$. *i.e.* $f$ is injective, $g$ is surjective and $\ker(f) = \mathrm{Im}(g)$.

A **short exact sequence** of cochain complexed is a sequence $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$ where $A^*, B^*, C^8$ are cochain complexes, $f, g$ are cochain maps, and $\forall k$, the sequence $0 \to A^k \xrightarrow{f} B^k \xrightarrow{g} C^k \to 0$ is exact.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{k-1} & \xrightarrow{f} & B^{k-1} & \xrightarrow{g} & C^{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{k} & \xrightarrow{f} & B^{k} & \xrightarrow{g} & C^{k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Given short exact sequence of cochain complexed $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$, we define a **boundary map** $\partial : H^k(C^*) \to H^{k+1}(A^*)$ as follows:

Take any $[c_0] \in H^k(C^*)$, $c_0 \in C^k$, $\mathrm{d}c_0 = 0$. $g$ is surjective so $\exists b_0 \in B^k \quad s.t.$ $g(b_0) = c_0$, and $g(\mathrm{d}b_0) = \mathrm{d}c_0 = 0$. So $\exists a_0 \in A^{k+1} \quad s.t.\ f(a_0) = \mathrm{d}b_0$, $f(\mathrm{d}a_0) = \mathrm{d}f(a_0) = \mathrm{d}^2 b_0 = 0$. Since $f$ is injective, $\mathrm{d}a_0 = 0$. Set $\partial[c_0] = [a_0]$.

$$
\begin{array}{ccccc}
 & & b_0 & \longrightarrow & c_0 \\
 & & \downarrow & & \downarrow \\
a_0 & \xrightarrow{f} & \mathrm{d}b_0 & \xrightarrow{g} & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{d}a_0 & \xrightarrow{f} & 0 & &
\end{array}
$$

**Lemma 7.13.** $\partial : H^k(C^*) \to H^{k+1}(A^*)$ *is well-defined*

*Proof.* If we choose another $b_1 \quad s.t.\ g(b_1) = g(b_0) = c_0$. Then

$$
\begin{aligned}
g(b_0 - b_1) = 0 &\Rightarrow b_0 - b_1 = f(a_2) \text{ for some } a_2 \in A^k \\
&\Rightarrow \mathrm{d}b_0 - \mathrm{d}b_1 = \mathrm{d}f(a_2) = f(\mathrm{d}a_2) \\
f(a_0) = \mathrm{d}b_0, f(a_1) = \mathrm{d}b_1 &\Rightarrow f(a_0 - a_1) = \mathrm{d}b_0 - \mathrm{d}b_1 = f(\mathrm{d}a_2) \\
&\Rightarrow a_0 - a_1 = \mathrm{d}a_2
\end{aligned}
$$

$$\Rightarrow [a_0] = [a_1] \in H^{k+1}(A^*)$$

If we choose $c_1$ with $[c_1] = [c_0] \in H^k(C^*)$. Then $\exists c_2 \in C^{k-1}$ *s.t.* $c_1 = c_0 + \mathrm{d}c_2$. Take any $b_2 \in B^{k-1}$ *s.t.* $g(b_2) = c_2$. Then $g(\mathrm{d}b_2) = \mathrm{d}c_2 = c_1 - c_0$. Pick $b_0$ *s.t.* $g(b_0) = c_0 \Rightarrow g(b_0 + \mathrm{d}b_2) = c_1$. So we may pick $b_1 = b_0 + \mathrm{d}b_2$. Then $\mathrm{d}b_1 = \mathrm{d}b_0 \Rightarrow a_1 = a_0$. □

**Lemma 7.14** (Snake Lemma)**.** *Given short exact sequence of cochain complexed* $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$*, the sequence*

$$\begin{array}{ccccccc}
H^{k+1}(C^*) & \xleftarrow{g^*} & H^{k+1}(B^*) & \xleftarrow{f^*} & H^{k+1}(A^*) & \leftarrow & \\
& & & \partial & & & \\
H^{k}(C^*) & \xleftarrow{g^*} & H^{k}(B^*) & \xleftarrow{f^*} & H^{k}(A^*) & \leftarrow & \\
& & & \partial & & & \\
H^{k-1}(C^*) & \xleftarrow{g^*} & H^{k-1}(B^*) & \xleftarrow{f^*} & H^{k-1}(A^*) & \leftarrow & \\
& & & \partial & & & \\
\cdots & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & 0
\end{array}$$

*is exact For simplicity, short exact sequence of cochain complexes induces long exact sequence on homotopy.*

*Proof.* **Exactness at** $H^k(B^*)$: $g \circ f = 0 \Rightarrow g^* \circ f^* = 0 \Rightarrow \mathrm{Im} f \subset \ker g^*$.

Take any $[b] \in \ker g^*$, $b \in B^*$, $\mathrm{d}b = 0$, then $[g(b)] = g^*([b]) = 0$. So $g(b) \in \mathrm{Im d}$ *i.e.* $\exists\, c \in C^{k-1}$ *s.t.* $\mathrm{d}c = g(b)$.

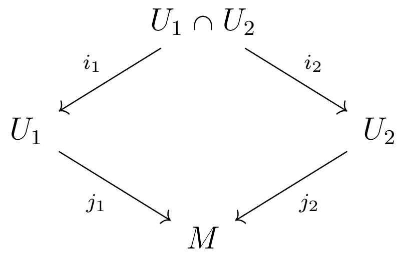
Pick $b' \in B^{k-1}$ *s.t.* $g(b') = c$. Then $[b] = [b - \mathrm{d}b']$. $g(b - \mathrm{d}b') = 0 \Rightarrow \exists a$ *s.t.* $f(a) = b - \mathrm{d}b'$. $f(\mathrm{d}a) = 0 \Rightarrow \mathrm{d}a = 0$. Then $f^*(a) = [b - \mathrm{d}b'] = [b]$. So $\ker g^* \subset \mathrm{Im} f^*$.

**Exactness at** $H^k(C^*)$: If $[c_0] \subset \ker(\partial)$. Then $[a_0] = 0 \in H^{k+1}(A^*)$ *i.e.* $\exists a_2 \in A^k$ *s.t.* $a_0 = \mathrm{d}a_2$. Set $b_2 = b_0 - f(a_2)$. Then $\mathrm{d}b_2 = \mathrm{d}b_0 - f(a_0) = 0$. So $[b_2] \in H^k(B^*)$, $g(b_2) = g(b_0) = c_0$. So $[c_0] = g^*([b_2]) \in \mathrm{Im} g^*$.

If $[c_0] \in \operatorname{Im}(g^*)$, then $\exists [b] \in H^k(B^*) \quad s.t.\ g^*[b] = [c_0]$. Set $c_1 = g(b)$, $b_1 = b$. Then $[c_1] = [c_0]$. Then $\mathrm{d}b_1 = 0$. So $a_1 = 0 \Rightarrow \partial[c_1] = \partial[c_0] = [a_1] = 0$. Then $\operatorname{Im}(g^*) \subset \ker(\partial)$.

**Exactness at** $\square$

Now we consider $M = U_1 \cup U_2$ for $U_1, U_2$ open subsets. The inclusion maps are $i_1, i_2, j_1, j_2$.




**Lemma 7.15.** *The sequence* $0 \to \Omega^*(M) \xrightarrow{(j_1^*, j_2^*)} \Omega^*(U_1) \bigoplus \Omega^*(U_2) \xrightarrow{i_1^* - i_2^*} \Omega^*(U_1 \cap U_2) \to 0$ *is exact.*

*Proof.* $(j_1^*, j_2^*)$ is injective. $\alpha \in \Omega^k(M)$. If $(j_1^*, j_2^*)(\alpha) = 0$, then $\alpha|_{U_1} = \alpha|_{U_2} = 0$. So $\alpha = 0$.

Exactness at $\Omega^*(U_1) \bigoplus \Omega^*(U_2)$: If $(\alpha_1, \alpha_2) \in \ker(i_1^* - i_2^*)$, then $\alpha_1|_{U_1 \cap U_2} = \alpha_2|_{U_1 \cap U_2}$. Set $\alpha \in \Omega^*(M) \quad s.t.\ \alpha|_{U_1} = \alpha_1, \alpha|_{U_2} = \alpha_2$. Then $(\alpha_1, \alpha_2) = (j_1^*\alpha, j_2^*\alpha)$.

If $(\alpha_1, \alpha_2) \in \operatorname{Im}(j_1^*, j_2^*)$. Let $\alpha_1 = \alpha|_{U_1}$, $\alpha_2 = \alpha|_{U_2}$, then $(i_1^* - i_2^*)(\alpha_1, \alpha_2) = \alpha|_{U_1 \cap U_2} - \alpha_{U_1 \cap U_2} = 0$. So $(\alpha_1, \alpha_2) \in \ker(i_1^* - i_2^*)$.

$i_1^* - i_2^*$ is surjective. Take any $\alpha_{12} \in \Omega^*(U_1 \cap U_2)$. Take a partition of unity $\{\rho_i : M \to [0, 1]\}_{i=1,2}$, $\rho_1 + \rho_2 \equiv 1$, $\operatorname{Supp}(\rho_0) \subset U_i$. Then $\operatorname{Supp}(\rho_i) \cap U_2 \subset U_1 \cap U_2$. So we may define $\alpha_i \in \Omega^*(U_i)$ by

$$\alpha_{i,p} = \begin{cases} (-1)^{i-1}\rho_i(p) \cdot \alpha_{12,p} & p \in U_1 \cap U_2 \\ 0 & p \in U_i \backslash (U_1 \cap U_2) \end{cases} \tag{7.12}$$

Then $\alpha_1|_{U_1 \cap U_2} - \alpha_2|_{U_1 \cap U_2} = \rho_1(p)\alpha_{12} + \rho_2(p)\alpha_{12} = \alpha_{12}$. $\square$

Apply the snake lemma 7.14, we obtain

**Theorem 7.16** (Mayer-Vietoris sequence)**.** *The sequence*

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & H^{k+1}_{dR}(U)\oplus H^{k+1}_{dR}(V) & \longleftarrow & H^{k+1}_{dR}(M) & \xleftarrow{d} & \\
H^{k}_{dR}(U\cap V) & \longleftarrow & H^{k}_{dR}(U)\oplus H^{k}_{dR}(V) & \longleftarrow & H^{k}_{dR}(M) & \xleftarrow{d} & \\
H^{k-1}_{dR}(U\cap V) & \longleftarrow & H^{k-1}_{dR}(U)\oplus H^{k-1}_{dR}(V) & \longleftarrow & \cdots & \longleftarrow & 0
\end{array}$$

*is exact*

**Example 7.17.** For $S^n = U\cap V$ where

$$U = S^n\backslash\{(1,0,\cdots,0)\}, V = S^n\backslash\{(-1,0,\cdots,0)\}, U\cap V = S^n\backslash\{(\pm1,0,\cdots,0)\}$$

$U,V\simeq *, U\cap V\simeq S^{n-1}$. So the M.V. sequence

$$0\longrightarrow H^0_{dR}(S^1)\longrightarrow H^0(*)\oplus H^0(*)\xrightarrow{\delta} H^0_{dR}(S^0)\longrightarrow H^1_{dR}(S^1)\longrightarrow 0$$

$$0\longrightarrow \mathbb{R}\longrightarrow \mathbb{R}\oplus\mathbb{R}\longrightarrow \mathbb{R}\oplus\mathbb{R}\longrightarrow \mathbb{R}\longrightarrow 0$$

Then $H^0_{dR}(S^1)=\ker\delta=\mathbb{R}$, $H^1_{dR}(S^1)=\mathrm{coker}\delta=\mathbb{R}$.

And for $k\geqslant 1$, the Mayer-sequence

$$0\longrightarrow H^k_{dR}(S^{n-1})\longrightarrow H^{k+1}_{dR}(S^n)\longrightarrow 0$$

implies $H^k_{dR}(S^{n-1})\cong H^{k+1}_{dR}(S^n)$. Therefore, by induction, we have

$$H^k_{dR}(S^n)=\begin{cases}\mathbb{R} & n=k\\ \mathbb{R} & k=0\\ 0 & \text{otherwise}\end{cases} \tag{7.13}$$

**Example 7.18.** For $M = \mathbb{CP}^n$, $U_1 = M\backslash\{[1,0,\cdots,0]\}$, $U_2 = M\backslash\{[0,*,*,\cdots,*]\}$. Then

$$\begin{aligned} U_1 &= \{[x_0,\cdots,x_n] | x_1,\cdots,x_n \text{ not all zero}\} \\ &\simeq \{[0,x_1,\cdots,x_n] | x_1,\cdots,x_n \text{ not all zero}\} \\ &\simeq \mathbb{CP}^{n-1} \end{aligned}$$

$$U_2 = M\backslash\{[0,*,\cdots,*]\} = \{[1,*,\cdots,*]\} \cong \mathbb{C}^n$$

$$U_1 \cap U_2 \cong U_2\backslash\{[1,0,\cdots,0]\} \cong \mathbb{C}^n\backslash\{0\} \simeq S^{2n-1}$$

*Claim.* $H^k_{dR}(\mathbb{CP}^n) \cong \begin{cases} \mathbb{R} & k \text{ even } 0 \leqslant k \leqslant 2n \\ 0 & \text{otherwise} \end{cases}.$

Prove the claim by induction. For $n = 1$, $\mathbb{CP}^1 \cong S^2$ is true.

Suppose we have proved it for $\mathbb{CP}^{n-1}$. Apply Mayer-Vietoris sequence to $M = U_1 \cup U_2$,

$$H^{i-1}_{dR}(S^{2n-1}) \to H^i_{dR}(\mathbb{CP}^n) \to H^i_{dR}(\mathbb{CP}^{n-1}) \oplus H^i_{dR}(\mathbb{C}^n) \to H^i_{dR}(S^{2n-1}) \tag{7.14}$$

Then for $i \neq 2n-1, 2n$,

$$0 \to H^i_{dR}(\mathbb{CP}^n) \xrightarrow{\cong} H^i_{dR}(\mathbb{CP}^{n-1}) \to 0 \tag{7.15}$$

For $2n-1, 2n$,

$$0 = H^{2n-1}_{dR}(\mathbb{CP}^{n-1}) \to H^{2n-1}_{dR}(S^{2n-1}) \xrightarrow{\cong} H^{2n}_{dR}(\mathbb{CP}^n) \to H^{2n}_{dR}(\mathbb{CP}^{n-1}) = 0 \tag{7.16}$$

$$0 = H^{2n-2}_{dR}(S^{2n-1}) \to H^{2n-1}_{dR}(\mathbb{CP}^n) \to H^{2n-1}_{dR}(\mathbb{CP}^{n-1}) = 0$$

**Remark 7.19.** $H^*(M)$ is a ring where

$$[\alpha]\cdot[\beta]=[\alpha\wedge\beta] \tag{7.17}$$

It's graded commutative since

$$[\alpha]\cdot[\beta]=(-1)^{kl}[\beta]\cdot[\alpha] \text{ for } [\alpha]\in H^k(M),\, \beta\in H^l(M) \tag{7.18}$$

Any $f:M\to N$ induces ring homomorphism $f^*:H^*_{dR}(N)\to H^*_{dR}(M)$.
Indeed, $H^*_{dR}(\mathbb{CP}^n)\cong \mathbb{R}[x]/_{(}x^{n+1})$, where $x$ generates $H^2_{dR}(\mathbb{CP}^n)$.

**Proposition 7.20.** *$A\subsetneq\mathbb{R}^n$ closed. Then*

$$H^{i+1}_{dR}(\mathbb{R}^{n+1}\backslash\{0\}\times A)\cong H^i_{dR}(\mathbb{R}^n\backslash A),\ i>0 \tag{7.19}$$

$$H^1_{dR}(\mathbb{R}^{n+1}\backslash\{0\}\times A)\cong H^0_{dR}(\mathbb{R}^n\backslash A)/_{\mathbb{R}\cdot 1} \tag{7.20}$$

*Proof.* $\mathbb{R}^{n+1}\backslash\{0\}\times A=U_1\cup U_2$ where

$$\begin{aligned} U_1 &= (\mathbb{R}_{<0}\times\mathbb{R}^n)\cup([0,1)\times(\mathbb{R}^n\backslash A)) \\ U_2 &= (\mathbb{R}_{>0}\times\mathbb{R}^n)\cup((-1,0]\times(\mathbb{R}^n\backslash A)) \\ U_1\cap U_2 &\cong (-1,1)\times(\mathbb{R}^n\times A) \end{aligned}$$

Then there exists a deformation retraction $[0,1]\times U_1\to U_1$, $(t,(x_0,x_1,\cdots,x_n))\mapsto (tx_0+(1-t)(-1),x_1,\cdots,x_n)$. Then $U_1\cong\{-1\}\times\mathbb{R}^n\simeq *$. $U_2\simeq *$, $U_1\cap U_2\simeq\mathbb{R}^n\backslash A$. The Mayer-Vietoris sequence gives

$$H^i_{dR}(U_1)\oplus H^i_{dR}(U_2)\to H^i_{dR}(U_1\cap U_2)\underset{\cong}{\xrightarrow{\partial}} H^{i+1}_{dR}(\mathbb{R}^{n+1}\backslash\{0\}\times A)\to H^{i+1}_{dR}(U_1)\oplus H^{i+1}_{dR}(U_2)$$

for $i>0$.

The case $i = 0$ is similar. □

**Lemma 7.21.** *$A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$ closed, $f : A \xrightarrow{\cong} B$. Then $\mathbb{R}^{n+m} \backslash \{0\} \times B \cong \mathbb{R}^{n+m} \backslash \{(x, f(x)) | x \in A\} \cong \mathbb{R}^{n+m} \backslash A \times \{0\}$.*

*Proof.* $A \underset{f}{\xrightarrow{\cong}} B \hookrightarrow \mathbb{R}^n$. By Tietz extension theorem, there exists $\widetilde{f} : \mathbb{R}^m \to \mathbb{R}^n$ *s.t.* $\widetilde{f}|_A = f$.

Consider $\widehat{f} : \mathbb{R}^m \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m \times \mathbb{R}^n$, $(x, y) \mapsto (x.y + \widehat{f}(x))$. Then $\widehat{f}(A \times \{0\}) = \{(x, f(x)) | x \in A\} \Rightarrow \mathbb{R}^{n+m} \backslash \{0\} \times B \cong \mathbb{R}^{n+m} \backslash \{(x, f(x)) | x \in A\} \cong \mathbb{R}^{n+m} \backslash A \times \{0\}$. □

Special case, if $A, B \subset \mathbb{R}^n$ closed, $A \cong B$, then $\mathbb{R}^{2n} \backslash A \times \{0\} \cong \mathbb{R}^{2n} \backslash B \times \{0\}$.

It directly follows that

**Theorem 7.22.** *$A, B \subset \mathbb{R}^n$ closed, $A \cong B$ under homeomorphism. $\Rightarrow H^*_{dR}(\mathbb{R}^n \backslash A) \cong H^*_{dR}(\mathbb{R}^n \backslash B)$.*

*Proof.* By proposition 7.20

$$H^*_{dR}(\mathbb{R}^n \backslash A) \cong H^{*+n}_{dR}(\mathbb{R}^{2n} \backslash A \times \{0\}) \cong H^{*+n}_{dR}(\mathbb{R}^{2n} \backslash B \times \{0\}) \cong H^*(\mathbb{R}^n \backslash B)$$

□

**Example 7.23.** A **knot** is an embedded $S^1 \hookrightarrow \mathbb{R}^3$. So $\forall$ knot $K$,

$$H^i_{dR}(\mathbb{R}^3 \backslash K) \cong H^i_{dR}(\mathbb{R}^3 \backslash \text{circle}) \cong \begin{cases} \mathbb{R} & i = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{7.21}$$

**Corollary 7.24.** *$A \subset \mathbb{R}^n$ closed, $A \cong S^{n-1} \Rightarrow \mathbb{R}^n \backslash A$ has two components $U_1, U_2$ where $U_1$ is bounded and $U_2$ is unbounded. Moreover, $\partial U_1 = \partial U_2 = A$.*

*Proof.* $H^0_{dR}(\mathbb{R}^n \backslash A) \cong H^0_{dR}(\mathbb{R}^n \backslash S^{n-1}) \cong \mathbb{R} \langle \pi_0(\mathbb{R}^n \backslash S^{n-1}) \rangle = \mathbb{R} \oplus \mathbb{R}$.

So $\mathbb{R}^n \backslash A$ has two components $U_1, U_2$.

Take $L = \max\{\|x\| | x \in A\} + 1$, $V = \{x \in \mathbb{R}^n | \|x\| > L\}$. $V \subset \mathbb{R}^n \backslash A$ connected and unbounded. $V \subset U_1$ so $U_1$ is unbounded $\Rightarrow U_2 \subset \mathbb{R}^n \backslash V$ is unbounded.

The proof of $\partial U_1 = \partial U_2 = A$ is omitted. See Madsen-Trnehave. □

**Corollary 7.25.** *$A \subset \mathbb{R}^n$, $A \subset D^k \Rightarrow \mathbb{R}^n \backslash A$ is connected.*

**Theorem 7.26** (Invariance of domain)**.** *Let $U$ be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}^n$ be continuous and injective map. Then $f(U)$ is also open in $\mathbb{R}^n$. And $f$ sends $U$ homeomorphically to $f(U)$*

*Proof.* It suffices to show $f(U)$ is open. Since for any $W \subset U$ open, $f(W)$ is open in $f(U)$. So $f|_U : U \to f(U)$ is open.

Take any $x_0 \in U$, want to show $f(x_0)$. Take $D = \{x \in \mathbb{R}^n | \|x - x_0\| \leqslant \delta\} \subset U$. Then $\Sigma = f(\partial D) \cong S^{n-1}$. So $\mathbb{R}^n \backslash \Sigma$ has 2 components $U_1, U_2$, where $U_1$ is bounded and $U_2$ is unbounded.

$\mathbb{R}^n \backslash f(D)$ is connected so $\mathbb{R}^n \backslash f(D) \subset U_2$. So $U_1 \cup \Sigma = \mathbb{R}^n \backslash U_2 \subset f(D) = f(\mathrm{int}(D)) \sqcup \Sigma \Rightarrow U_1 \subset f(\mathrm{int}(D))$.

Since $f(\mathrm{int}(D))$ is connected, $f(\mathrm{int}(D)) \subset U \Rightarrow f(\mathrm{int}(D)) = U_1$. So $f(x_0) \in U_1 \subset \mathrm{int} f(U)$. □

**Corollary 7.27.** *If $m > n$, $U \subset \mathbb{R}^m$ open, then there is no injective continuous map $U \to \mathbb{R}^n$*

*Proof.* If $f : U \to \mathbb{R}^n \hookrightarrow \mathbb{R}^m$. Then $f(U)$ not open in $\mathbb{R}^m$, which causes contradiction. □

**Corollary 7.28.** *$U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^m$ open. $U \cong V \Rightarrow m = n$.*

## 7.4 Compact supported de Rham cohomology

Define $H^*_{dR,c}(M) = H^*(\Omega^0(M) + \Omega^1(M) + \cdots)$, abbreviated to $H^*_c(M)$.

If $M$ is compact, then $H^*_{dR,c}(M) \cong H^*_{dR}(M)$. But it is not true for $M$ not compact. The following is a counterexample.

**Theorem 7.29.** *Let $M$ be connected and oriented $n$-dimentional manifold without boundary. Then the map $H^n_c(M) \xrightarrow{\cong} \mathbb{R}, [\alpha] \mapsto \int_M \alpha$ is a well-defined isomorphism. Moreover, if $M$ is closed and connected, then $H^n_{dR}(M) \cong \mathbb{R}$*

**Remark 7.30.** Well-defineness: If $[\alpha] = [\alpha']$, then $\alpha - \alpha' = \mathrm{d}\beta$. By Stokes theorem 6.12, $\int_M \alpha = \int_M \alpha'$.

**Proposition 7.31.** *If $\alpha \in \Omega^n_c(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \alpha = 0 \Rightarrow \exists \beta \in \Omega^{n-1}_c(\mathbb{R}^n)$ s.t. $\mathrm{d}\beta = \alpha$.*

Note that if $\alpha = f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, $\beta = \sum_{i=1}^{n}(-1)^{i-1} f_i \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$. Then $\alpha = \mathrm{d}\beta$ iff $f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$.

So the proposition is equivalent to $\forall f \in C^\infty_c(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = 0 \Rightarrow \exists f_1, \cdots, f_n \in C^\infty_c(\mathbb{R}^n)$ s.t. $f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$.

Prove it by induction. For $n = 1$, $f_1(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. $f_1 \in C^\infty_c(\mathbb{R})$, $f_1' = f$.

Suppose $n$ is proved, $f \in C^\infty_c(\mathbb{R}^{n+1})$, $\int_{\mathbb{R}^{n+1}} f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1} = 0$.

Define $g \in C^\infty_c(\mathbb{R}^n)$ by $g(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} f(x_1, \cdots, x_{n+1})\mathrm{d}x_{n+1}$. Then $\int_{\mathbb{R}^n} g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = 0$.

By induction, $\exists\, g_1, \cdots, g_n \in C^\infty_c(\mathbb{R}^n)$ s.t. $g = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}$.

Take $\rho \in C^\infty_c(\mathbb{R})$ s.t. $\int_{-\infty}^{\infty} \rho\mathrm{d}x = 0$. Define $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ by $f_i(x_1, \cdots, x_{n+1}) = g_i(x_1, \cdots, x_n) \cdot \rho(x_{n+1})$.

Set $h = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$. Then for $\forall (x_1, \cdots, x_n)$,

$$\int_{-\infty}^{\infty} h(x_1, \cdots, x_{n+1})\mathrm{d}x_{n+1} = \int_{-\infty}^{\infty} f(x_1, \cdots, x_{n+1}) - \int_{-\infty}^{\infty} \rho(x_{n+1})\mathrm{d}x_{n+1} \cdot \left(\sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}\right) = 0$$

Set $f_{n+1}(x_1, \cdots, x_{n+1}) = \int_{-\infty}^{x_{n+1}} h(x_1, \cdots, x_n, t)\mathrm{d}t$. Then $\frac{\partial f_{n+1}}{\partial x_{n+1}} = h$ and $f = \sum_{i=1}^{n+1} \frac{\partial f_i}{\partial x_i}$.

**Corollary 7.32.** *$H_c^n(\mathbb{R}^n) \xrightarrow{\cong} \mathbb{R}, [\alpha] \mapsto \int_{\mathbb{R}^n} \alpha$ is an isomorphism.*

**Lemma 7.33.** *$\forall \alpha \in \Omega_c^n(\mathbb{R}^n), \forall U \subset \mathbb{R}^n$ open, $\exists \beta \in \Omega_c^{n-1}(\mathbb{R}^n) \quad s.t. \ \mathrm{Supp}(\alpha - \mathrm{d}\beta) \subset U$*

*Proof.* Pick $\alpha' \in \Omega_c^n(U) \subset \Omega_c^n(\mathbb{R}^n) \quad s.t. \ \int_U \alpha' = \int_{\mathbb{R}^n} \alpha$.

Then $\exists \beta \in \Omega_c^{n-1}(\mathbb{R}^n) \quad s.t. \ \alpha - \alpha' = \mathrm{d}\beta$. So $\mathrm{Supp}(\alpha - \mathrm{d}\beta) = \mathrm{Supp}(\alpha') \subset U$. ☐

This lemma tells us every $\alpha \in H_c^n(\mathbb{R})$ can be restricted to $H_c^n(U)$.

**Lemma 7.34.** *$M$ connected. Given charts $U_1 \xrightarrow{\varphi_1} \mathbb{R}^n, U_2 \xrightarrow{\varphi_2} \mathbb{R}^n$, $\alpha_1 \in \Omega_c^n(U_1)$. $\exists \beta \in \Omega_c^{n-1}(M) \quad s.t. \ \mathrm{Supp}(\alpha_1 - \mathrm{d}\beta) \subset U_2$.*

This lemma tells us every $n$-form with compact support in a chart can be transformed to a $n$-form with compact support in the next chart.

*Proof.* We can find charts $\{V_i \cong \mathbb{R}^n\}_{1 \leqslant i \leqslant k} \quad s.t. \ V_1 = U_1, V_k = U_2, V_i \cap V_{i+1} \neq \varnothing$.

By previous lemma, $\exists \beta_i \in \Omega_c^{n-1}(V_i) \quad s.t. \ \mathrm{Supp}(\alpha_i - \mathrm{d}\beta_i) \subset V_i \cap V_{i+1} \subset V_2$. Define $\alpha_{i+1} = \alpha_i - \mathrm{d}\beta_i \in \Omega_c^n(V_{i+1})$.

Then we obtain $\beta_1, \cdots, \beta_{n-1} \quad s.t. \ \mathrm{Supp}(\alpha_1 - \sum_{i=1}^{n-1} \mathrm{d}\beta_i) \subset V_k = U_2$. ☐

*Proof of Theorem 7.29.* We have proved the well-definedness by Stokes formula and sujevectivity straight forward.

Injevtivity: Given $\alpha \in \Omega_c^n(M)$, $\int_M \alpha = 0$. Using partition of unity, decompose $\alpha = \alpha_1 + \cdots + \alpha_k \quad s.t. \ \mathrm{Supp}(\alpha_i)$ compact, $\mathrm{Supp}(\alpha_i) \subset U_i \cong \mathbb{R}^n$.

By previous lemma, $\forall 2 \leqslant i \leqslant k$, $\exists \beta_i \in \Omega_c^{n-1}(M) \quad s.t. \ \mathrm{Supp}(\alpha_i - \mathrm{d}\beta_i) \subset U_1$.

Then $\alpha' = \alpha - \sum_{i=2}^{k} \mathrm{d}\beta_i \in \Omega_c^n(U_1)$. $\int_{U_1} \alpha' = \int_M \alpha = 0$.

So $\exists \beta_1 \in \Omega_c^n(U_1) \subset \Omega_c^n(M) \quad s.t. \ \alpha' = \mathrm{d}\beta_1$. Set $\beta = \sum_{i=1}^{n} \beta_i$. Then $\alpha = \mathrm{d}\beta$. ☐

Given $n$-dimensional connected, oriented, closed manifold $M, N$, $f : M \to N$. The mapping degree of $f$ is defined by

$$\mathbb{R} \overset{\int_N}{\cong} H^n(N) \xrightarrow{f^*} H^n(M) \overset{\int_M}{\cong} \mathbb{R}, 1 \mapsto \deg(f)$$

Equivalently, $\forall\alpha \in \Omega^n(N)$, we have $\int_M f^*(\alpha) = \deg(f)\int_N \alpha$.

Then $\deg(f)$ is invariant under homotopy.

If $M = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_k$, then define $\deg(f : M \to N) := \sum_{i=1}^{k}\sum_{i=1}^{k}\deg(f|_{M_i} : M_i \to N)$.

There is an alternative definition. For $f : M \to N$, take a regular value $y$ of $f$, *i.e.* $\forall x \in f^{-1}(y), f_* : T_xM \to T_yN$ is surjective. Then $\forall x \in f^{-1}(U)$, $\exists$ neighbourhood $U_x$ of $x$, $V_y$ of $y$ *s.t.* $f|_{U_x} : U_x \xrightarrow{\cong} V_y$ by inverse function theorem. In particular, $f^{-1}(y)$ is discrete. $M$ is compact implies that $f^{-1}(y) = \{x_1, \cdots, x_k\}$.

Define local degree $\deg(f, x_i) = \begin{cases} 1 & \text{if } f_* : T_{x_i}M \xrightarrow{\cong} T_yN \text{ is orientation preserving} \\ -1 & \text{otherwise} \end{cases}$

**Theorem 7.35.** *$\forall$ regular value $y$ of $f$,* $\deg(f) = \sum_{x\in f^{-1}(y)} \deg(f, x) \in \mathbb{Z}$

*Proof.* For $f^{-1}(y) = \{x_1, \cdots, x_n\}$, $\exists$ neighborhood $U_i$ of $x_i$, $V_i$ of $y$ *s.t.* $f|_{U_i} : U_i \xrightarrow{\cong} V_i$. Let $V$ neighborhood of $y$ *s.t.* $f^{-1}(V) \subset \bigcup_{i=1}^{k} U_i$.

Take $\alpha \in \Omega^n(N)$ *s.t.* $\mathrm{Supp}(\alpha) \subset V\bigcap(\bigcap_{i=1}^{k} V_i)$, $\int_N \alpha \neq 0$, then $\mathrm{Supp}(f^*\alpha) \subset f^{-1}(V) \subset \bigcup_{i=1}^{k} U_i$.

$\int_M f^*(\alpha) = \sum_{i=1}^{k}\int_{U_i} f^*\alpha = \sum_{i=1}^{k}\deg(f, x_i)\cdot\int_N \alpha$ since

$$\deg(f, x_i) = 1 \Leftrightarrow f|_{U_i} \text{ is oreientation preserving} \Leftrightarrow \int_{U_i} f^*\alpha = \int_{V_i}\alpha = \int_N \alpha$$

$$\deg(f, x_i) = -1 \Leftrightarrow f|_{U_i} \text{ is oreientation reversing} \Leftrightarrow \int_{U_i} f^*\alpha = -\int_{V_i}\alpha = -\int_N \alpha$$

So $\deg(f) = \sum_{i=1}^{k}\deg(f, x_i)$. ☐

**Corollary 7.36.** *If $f$ is not surjective (after homotopy) then* $\deg(f) = 0$

For $X \in \Gamma(TM)$, $p \in M$ is an isolated singular point. Then $\exists$ open $U \xrightarrow[\cong]{\varphi} \mathbb{R}^n, p \mapsto \vec{0}$ of $p$ s.t. $X_q \neq 0 \ \forall q \in U \backslash \{p\}$. Define the local index of $X$ at $p$ by

$$(X, p) = \deg\left(\frac{\varphi_*(X|_U)}{|\varphi_*(X|_U)} : S^{n-1} \to S^{n-1}\right) \in \mathbb{Z}$$

where $\varphi_*(X|_U) \in \Gamma(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ maps $0$ to $0$.

**Example 7.37.** $M = \mathbb{R}^2$, $X = x\partial x + y\partial y$. Then $(X, 0) = 1$.

**Theorem 7.38** (Poincáre-Hopf). *Let $M$ be oriented closed manifold. $X$ vector field with only isolated singularity. Then $\sum_{\{p|X_p=0\}} (X, p) = \chi(M)$ is the Euler characteristic of $M$, i.e. $\sum_{k=0}^{\dim(M)} (-1)^k b_k$, $b_k = \dim_{\mathbb{R}} H^k_{dR}(M)$.*

**Theorem 7.39** (Poincáre duality theorem). *$M$ is oriented $n$-dimensional manifold withour boundary. Then the bilinear map $H^k(M) \times H^{n-k}_c(M) \to \mathbb{R}, ([\alpha], [\beta]) \mapsto \int_C \alpha \wedge \beta$ induces $D_M : H^k(M) \to \left(H^{n-k}_c(M)\right)^*$ which is always an isomorphism.*

**Proposition 7.40.** $H^k_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases}$

*Proof.* We have proved for $k = n$. For $k = 0$, $\alpha \in \Omega^0_c(\mathbb{R}^n)$, $\mathrm{d}\alpha = 0 \Rightarrow \alpha = 0$.

For $1 < k < n$, $\alpha \in \Omega^k_c(\mathbb{R}^n)$, $\mathrm{d}\alpha = 0$. We want to find $\beta \in \Omega^{k-1}_c(\mathbb{R}^n)$ s.t. $\alpha = \mathrm{d}\beta$.

$\mathbb{R}^n = S^n \backslash \{\infty\}$. Regard $\alpha$ as an element in $\Omega^k(S^n)$, $\mathrm{d}\alpha = 0$. $H^k(S^n) = 0 \Rightarrow \exists \beta' \in \Omega^{k-1}(S^n), \mathrm{d}\beta' = \alpha$.

Take open neighborhood $U \cong \mathbb{R}^n$ of $\{\infty\} \in S^n$ s.t. $\alpha|_U = 0 \Rightarrow \mathrm{d}(\beta'|_U) = 0$. If $k = 1$, $\beta'|_U \equiv C$. Set $\beta = \beta' - C$, $\beta|_U \equiv 0$ so $\beta \in \Omega^{k-1}_c(\mathbb{R}^n)$. If $k > 1$, $H^{k-1}(U) = 0 \Rightarrow \exists \gamma \in \Omega^{k-2}(U)$ s.t. $\beta|_U = \mathrm{d}\gamma$. Extend $\gamma$ to $\gamma' \in \Omega^{k-2}(S^n)$. Set $\beta = \beta' - \mathrm{d}\gamma'$. Then $\beta|_U = \beta'|_U - \mathrm{d}\gamma = 0$ so $\beta \in \Omega^{k-1}_c(\mathbb{R}^n)$ and $\mathrm{d}\beta = \mathrm{d}\beta' - \mathrm{d}^2\gamma' = \alpha$. ☐

So we prove that $D_M$ is an isomorphism when $M = \mathbb{R}^n$.

For $U \overset{i}{\hookrightarrow} V$, $U, V$ open in $M$. The map $\Omega_c^*(U) \xrightarrow{i_*} \Omega_c^*(V), \alpha \mapsto i_*(\alpha)_p = \begin{cases} \alpha_p & p \in U \\ 0 & p \notin U \end{cases}$

**Lemma 7.41.** *The sequence*

$$0 \to \Omega_c^*(U_1 \cap U_2) \xrightarrow{(i_{1,*}, i_{2,*})} \Omega_c^*(U_1) \oplus \Omega_c^*(U_2) \xrightarrow{j_1^* - j_2^*} \Omega_c^*(U_1 \cup U_2) \to 0$$

*is exact. Moreover, by snake lemma 7.14, we have M.V. sequence for $H_c^*(-)$:*

$$H_c^*(U_1 \cap U_2) \to H_c^*(U_1) \oplus H_c^*(U_2) \to H_c^*(U_1 \cup U_2) \xrightarrow{\partial} H_c^{k+1}(U_1 \cap U_2) \to \cdots$$

*Take dual, we get*

$$\begin{array}{ccccccccccc} \cdots \longleftarrow & (H_c^k(U_1 \cap U_2))^* & \longleftarrow & (H_c^k(U_1))^* \oplus (H_c^k(U_2))^* & \longleftarrow & (H_c^k(U_1 \cap U_2))^* & \longleftarrow & \left(H_c^{k+1}(U_1 \cap U_2)\right)^* & \longleftarrow \cdots \\ & \uparrow{\scriptstyle D_{U_1 \cap U_2}} & & \uparrow{\scriptstyle D_{U_1} \oplus D_{U_2}} & & \uparrow{\scriptstyle D_{U_1 \cup U_2}} & & \uparrow{\scriptstyle D_{U_1 \cap U_2}} & \\ \cdots \longleftarrow & H^{n-k}(U_1 \cap U_2) & \longleftarrow & H^{n-k}(U_1) \oplus H^{n-k}(U_2) & \longleftarrow & H^{n-k}(U_1 \cup U_2) & \underset{\partial}{\longleftarrow} & H_c^{n-k-1}(U_1 \cap U_2) & \longleftarrow \cdots \end{array}$$

**Lemma 7.42.** *The diagram commutes.*

**Lemma 7.43** (The five lemma)**.** *If the diagram commutes, $f_1, f_2, f_4, f_5$ are isomorphisms, then $f_3$ is an isomorphism.*

$$\begin{array}{ccccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & & \textit{exact} \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_5} & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 & & \textit{exact} \end{array}$$

**Lemma 7.44.** $\forall M, \exists$ *locally finite open cover* $\mathcal{U} = \{U_i\}_{i \in I}$ *s.t.* $\forall J \subset I, \bigcap_{i \in J} U_i \cong \begin{cases} \varnothing \\ \mathbb{R}^n \end{cases}$.

# 8 Classical Differential Geometry

## 8.1 The geometry of curves and surfaces

A **parametrized curve** is a smooth map $\alpha : I = (a,b) \to \mathbb{R}^3$. $t(s) = \alpha'(s)$ is called the **tangent vector** at $\alpha(s)$.

A reparametrization of $\alpha$ means $(a',b') \xrightarrow{\cong} (a,b) \xrightarrow{\alpha} \mathbb{R}^3$.

We say $\alpha$ is a **regular** curve if $t(s) \neq 0$, $\forall s \in I$.

We say $\alpha$ is **parametrized by its arc length** if $|t(s)| = 1$, $\forall s \in I$. (It can be obtained by reparametrization)

Always assume $\alpha$ is parametrized by its arc length.

Define $K(s)= |\alpha''(s)| \in \mathbb{R}_{\geqslant 0}$, the **curvature** of $\alpha$ at $\alpha(s)$.

If $K(s) \neq 0$, we may define the **normal vector** $n(s) = \dfrac{\alpha''(s)}{K(s)}$. Then $0 = \dfrac{\mathrm{d}}{\mathrm{d}s}\langle \alpha'(s), \alpha'(s)\rangle = 2\langle \alpha''(s), \alpha'(s)\rangle$. $\Rightarrow n(s) \perp t(s)$. The **osculating plane** of $\alpha$ at $\alpha(s)$ is $\mathrm{Span}\,\langle t(s), n(s)\rangle$. And **binormal vector** $b(s) = t(s) \times n(s)$. Then $t(s), n(s), b(s)$ is an orthogonormal basis of $T_{\alpha(s)}\mathbb{R}^3$. $b'(s) = t'(s) \times n(s) + t(s) \times n'(s) \perp t(s), b(s)$. So $b'(s) = \tau(s) \cdot n(s)$ for some $\tau : I \to \mathbb{R}$. $\tau(s)$ is called the **torsion** of $\alpha$ at $\alpha(s)$.

**Proposition 8.1** (Frenet formula). $\begin{cases} t' = Kn \\ n' = -Kt - \tau b \\ b' = \tau n \end{cases}$.

*Proof.* $n'(s) = (b(s) \times t(s))' = \tau(s)n(s) \times t(s) + b(s) \times (K(s) \cdot n(s)) = -\tau(s)b(s) - K(s)t(s)$. The other two are straight forward. □

**Theorem 8.2.** *Given any smooth function $K : I \to \mathbb{R}_{>0}$, $\tau : I \to \mathbb{R}$, there exists a curve $\alpha$ parametrized by its arc length $s$ s.t. $K(s)$ is its curvature and $\tau(s)$ is its torsion. Any two such curves differ by a rotation and a translation in $\mathbb{R}^3$. i.e. $\forall \alpha_1, \alpha_2$, $\exists \rho \in \mathrm{SO}(3)$, $c \in \mathbb{R}^3$ s.t. $\alpha_2 = \rho \circ \alpha_1 + C$.*

*Proof.* Uniqueness: After rotation and translation, one can assume $\alpha_1(0) = \alpha_2(0)$, $t_1(0) = t_2(0)$, $n_1(0) = n_2(0)$, $b_1(0) = b_2(0)$. By the uniqueness of ODE, $t_1(s) = t_2(s)$, $\forall s \in I$. Hence $\alpha_1 = \alpha_2$.

Existence: Solve the Frenet formula 8.1 with any initial $(t(0), n(0), b(0))$. We get a local solution $t(s), n(s), b(s) : I' \to \mathbb{R}^3$ with maximal domain. Want to show $I' = I$. Just need to show $|t(s)|$, $|n(s)|$, $|b(s)|$ are bounded when $s \in K \subset I$ in a compact set $K$.

Then $\exists A \in \mathbb{R}_{>0}$ *s.t.* $|K(s)|, |\tau(s)| \leqslant A$, $\forall s \in K$.

$$\frac{\mathrm{d}}{\mathrm{d}s}(|t(s)| + |n(s)| + |b(s)|) \leqslant |t'(s)| + |n'(s)| + |b'(s)| \leqslant 2A(|t(s) + |n(s)| + |b(s)|)$$

$$\Rightarrow |t(s)| + |n(s)| + |b(s)| \leqslant e^{2As} \cdot (|t(0)| + |n(0)| + |b(0)|)$$

So $I' = I$ by continuity.

Let $\alpha(s) = \alpha'(0) + \int_0^s t(s)\mathrm{d}x$. Then $\alpha$ has the given curvature and torsion. $\square$

## 8.2 Theory of surfaces in $\mathbb{R}^3$

A smooth embedded surface $S \overset{\iota}{\hookrightarrow} \mathbb{R}^3$ is called a **regular surface**.

We identify $S$ with $\iota(s)$. Then $T_pS \subset T_p\mathbb{R}^3 = \mathbb{R}^3$, $\forall p \in S$.

$\forall p \in S$, we pick local chart $X : V \to U \overset{\text{open}}{\subset} S \hookrightarrow \mathbb{R}^3, (u,v)^T \mapsto (x(u,v), y(u,v), z(u,v))^T$, called a **local parametrization** of $S$.

$X_u := X_*(\frac{\partial}{\partial u}) = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})^T$.

$X_v := X_*(\frac{\partial}{\partial v}) = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})^T$.

The standard restrction on $\mathbb{R}^3$ restricts to a Riemann matrix on $S$ *i.e.* $\forall p \in S$, we have a symmetric, positive definite bilinear form $\langle -, - \rangle_p : T_pS \otimes T_pS \to \mathbb{R}, \langle \omega_1, \omega_2 \rangle_p = \langle \omega_1, \omega_2 \rangle_{\mathbb{R}^3}$.

$\langle -, - \rangle$ is determined by the quadratic form $\mathrm{I}_p : T_pS \to \mathbb{R}, \vec{v} \mapsto |\vec{v}|^2$ called the

**1st fundamental form**

**Example 8.3.** $S = \text{graph}(f) = \{(u, v, f(u, v))\}$. Then

$$X_u = \begin{pmatrix} 1 \\ 0 \\ \dfrac{\partial f}{\partial u} \end{pmatrix}, \; X_v = \begin{pmatrix} 0 \\ 1 \\ \dfrac{\partial f}{\partial v} \end{pmatrix}$$

So $\mathrm{I}_p(aX_u + bX_v) = a^2 + b^2 + (a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v})^2$

Recall that the induced volume form on $\mathbb{R}^2$ is $\mathrm{dVol} = \sqrt{\det(g)}\mathrm{d}u\mathrm{d}v = |X_u \times X_v|\mathrm{d}u\mathrm{d}v$.

## 8.3 Gauss map

From now on, assume $S$ is oriented, pick oriented local parametrization $X$. Then $\forall p \in S$, $\exists N(p) \in \mathbb{R}^3$ *s.t.* $|N(p)| = 1$, $N(p) \perp T_pS$, $(X_u, X_v, N(p))$ oriented basis of $\mathbb{R}^3$. $N(p)$ is called the **normal vector** of $S$ at $p$. **Gauss map** is $N : S \to S^2, p \mapsto N(p)$.

Take the differential at $p$, $\mathrm{d}N_p = N_{*,p} : T_pS \to T_{N(p)}S^2 = N(p)^\perp = T_pS$.

**Proposition 8.4.** *$\mathrm{d}N_p : T_pS \to T_pS$ is symmetric or equivalently self-adjoint. i.e.*

$$\langle \mathrm{d}N_p(\omega_1), \omega_2 \rangle_p = \langle \omega_1, \mathrm{d}N_p(\omega_2) \rangle_p, \; \forall \omega_2 \in T_pS$$

*Proof.* Take local parametrization $X : V \to S \subset \mathbb{R}^3$, $(u, v) \mapsto p$. By linearity, it suffices to check $\omega_1 = X_u$ and $\omega_2 = X_v$.

$\langle N(u, v), X_u \rangle = 0 \Rightarrow \left\langle \dfrac{\partial N}{\partial v}, X_{uv} \right\rangle + \langle N(u, v), X_{uv} \rangle = 0$. So $\langle \mathrm{d}N_p(X_v), X_u \rangle = - \langle N(u, v), X_{uv} \rangle = \langle X_u, \mathrm{d}N_p(X_u) \rangle$ □

**Remark 8.5.** Here we actually obtain

$$-\langle N_v, X_u\rangle = \langle N, X_{u,v}\rangle$$

Similarly, one can prove

$$-\langle N_u, X_v\rangle = \langle N, X_{v,u}\rangle$$

$$-\langle N_u, X_u\rangle = \langle N, X_{u,u}\rangle$$

$$-\langle N_v, X_v\rangle = \langle N, X_{v,v}\rangle$$

Define the quadratic form $\mathrm{II}_p : T_pS \to \mathbb{R}$, $\omega \mapsto -\langle \mathrm{d}N_p(\omega), \omega\rangle$. The induced form is called the **2nd fundamental form**

**Definition 8.6.** Given $p \in S$, a curve $\alpha : I \to S \subset \mathbb{R}^3, 0 \mapsto p$. Let $n_p$ be the normal vector of $\alpha$ at $p$. Let $\theta$ be the angle between $n_p$ and $N(p)$. $K$ is the curvature of $\alpha$ at $p$. $K_n := K \cdot \cos\theta$ is called the **normal curvature**. Then if $\alpha$ is parametrized by arc length, $K_n = \langle N(p), \alpha''(s)\rangle$. Since $\langle N(\alpha(s)), \alpha'(s)\rangle = 0 \Rightarrow$

$$\langle \mathrm{d}N_p(\alpha'(0)), \alpha'(0)\rangle + \langle N(\alpha(0)), \alpha''(0)\rangle = 0 \Rightarrow K_n = -\langle \mathrm{d}N_p(\alpha'(0)), \alpha'(0)\rangle = \mathrm{II}_p(\alpha'(0))$$

**Theorem 8.7** (Meusnier)**.** *All curves in $S$ through $p \in S$, with same tangent vector $v \in T_pS$ at $p$, $|v| = 1$, have the same normal curvature $K_n = \mathrm{II}_p(v)$. In particular, $L_v = \mathrm{Span}\langle N_p, v\rangle$, $\alpha_v = L \cap S$. Then for $\alpha_v$, $K_n = \pm K$ at $p = \mathrm{II}_p(v)$.*

*So we call $\mathrm{II}_p(v)$ the normal curvature of $S$ in the direction of $v \in T_pS$.*

**Definition 8.8.** $\mathrm{d}N_p$ has eigenvector $e_1, e_2$ and

$$\mathrm{d}N_p(e_1) = -k_1 e_1,\ \mathrm{d}N_p(e_2) = -k_2 e_2$$

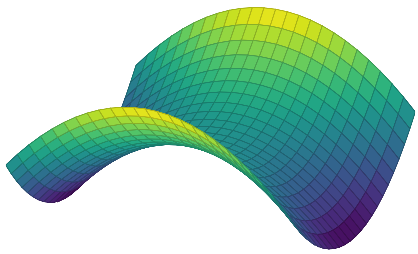
Then $k_1, k_2$ is called the **principal curvature** of $S$ at $p$, $e_1, e_2$ is called the **principal**

**directions** of $S$ at $p$.

**Definition 8.9.** If $k_1, k_2$ is the principal curvatures of $S$ at $p$, $N_p$ is the Gauss map, then the **Gaussian curvature** is $K = \det \mathrm{d}N_p = k_1 k_2$. The **average curvature** $H = -\frac{1}{2}\mathrm{trd}N_p = \frac{1}{2}(k_1 + k_2)$.

**Example 8.10.** For a cylinder $\{(x, y, z) | x^2 + y^2 = 1\}$, $N(p) = (-x, -y, 0)$, $\mathrm{d}N_p = (-x'(p), -y'(p), 0)$. So $k_1 = 0, k_2 = 1, K = 0$.

**Example 8.11.** $S = \{z = y^2 - x^2\} \subset \mathbb{R}^2$. $\mathrm{d}N = \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}(2u, -2v, 1)$. Then $k_1 = 2, k_2 = -2, K = -4 < 0$.



**Definition 8.12.** For $p \in S$ in a surface,

Call it **elliptic** if $\det \mathrm{d}N_p > 0$.

Call it **hyperbolic** if $\det \mathrm{d}N_p < 0$.

Call it **parabolic** if $\det \mathrm{d}N_p = 0$ but $\mathrm{d}N_p \neq 0$, or call it **planar** if $\mathrm{d}N_p = 0$.

The 1st fundamental form $\mathrm{I}_p = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$ where

$$E = \langle X_u, X_u \rangle \, , \, F = \langle X_u, X_v \rangle \, , \, G = \langle X_v, X_v \rangle \tag{8.1}$$

However, $\mathrm{II}(\alpha') = e\mathrm{d}u^2 + 2f\mathrm{d}u\mathrm{d}v + g\mathrm{d}v^2$ where

$$e = \langle N, X_{u,u} \rangle \, , \, f = \langle N, X_{u,v} \rangle \, , \, g = \langle N, X_{v,v} \rangle \tag{8.2}$$

Since $\{X_u, X_v, N\}$ is an orthogonormal basis in $\mathbb{R}^3$, $\langle N, N\rangle = 1 \Rightarrow 2\langle N_u, N\rangle = 0$ if we take $\dfrac{\partial}{\partial u}$. Then

$$\begin{cases} N_u = a_{11}X_u + a_{21}X_v \\ N_v = a_{12}X_u + a_{22}X_v \end{cases}$$

Then for $\mathrm{d}N(\alpha') = \mathrm{d}N(X_u \cdot u' + X_v \cdot v') = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}$. Therefore, $\mathrm{d}N = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Now

$$f = \langle N, X_{u,v}\rangle = -\langle N_v, X_u\rangle = -\langle a_{12}X_u + a_{22}X_v, X_u\rangle = -a_{12}E - a_{22}F$$

Similarly, one can prove

$$f = -\langle N_v, X_u\rangle = -a_{12}E - a_{22}F$$

$$e = -\langle N_u, X_u\rangle = -a_{11}E - a_{21}F$$

$$g = -\langle N_v, X_v\rangle = -a_{12}F - a_{22}G$$

So

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\frac{1}{EG - F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \tag{8.3}$$

*i.e.*

$$\begin{aligned} a_{11} &= \frac{fF - eG}{EG - F^2} \quad a_{12} = \frac{gF - fG}{EG - F^2} \\ a_{21} &= \frac{eF - fE}{EG - F^2} \quad a_{22} = \frac{fE - gE}{EG - F^2} \end{aligned} \tag{8.4}$$

So $K = \det \mathrm{d}N_p = \dfrac{eg - f^2}{EG - F^2}$, $H = -\mathrm{trd}N_p = \dfrac{1}{2} \cdot \dfrac{eG - 2fF + gE}{EG - F^2}$.

**Example 8.13.** For torus $T^2 = S^1 \times S^1$,

$X(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$.

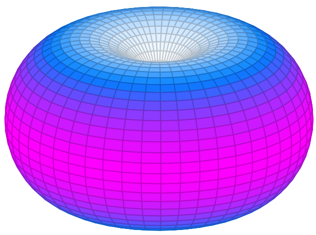
Then we can calculate $E, F, G, N$ by calculating $X_u, X_v$. And we obtain $f, g, e$ in a similar way. The result

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\cos u}{r(a + r\cos v)}$$

Then

$$\begin{cases} K = 0 & u = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \\ K < 0 & \dfrac{3\pi}{2} > u > \dfrac{\pi}{2} \\ K > 0 & \text{otherwise} \end{cases}$$

So the torus is elliptic in the outer half and is hyperbolic in the inner half.



**Theorem 8.14** (Gauss Theorem EGREGIUM). *$K$ is the invariant of $S$, only depending on $E, F, G$. Equivalently, Gauss curvature is invariant under local isometry.*

*Proof.* For $\{X_u, X_v, N\}$ an orthogonormal basis of $\mathbb{R}^3$, let

$$X_{u,u} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L_1 N$$

$$X_{u,v} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + L_2 N$$

$$X_{v,u} = \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + \widetilde{L}_2 N$$

$$X_{v,v} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + L_3 N$$

where $\Gamma_{ij}^k$ is called **Cristopher symbol**, $\Gamma_{21}^i = \Gamma_{21}^i, i = 1, 2$

$$N_u = a_{11} X_u + a_{12} X_v$$

$$N_v = a_{12} X_u + a_{22} X_v$$

Then take the inner product of $X_{u,u}$ with $N, X_u, X_v$ separately

$$e = \langle X_{u,u}, N \rangle = L_1$$

$$\frac{1}{2} E_u = \frac{1}{2} \cdot \frac{\partial}{\partial u} \langle X_u, X_u \rangle = \langle X_{u,u}, X_u \rangle = \Gamma_{1,1}^1 E + \Gamma_{12}^2 F$$

$$F_u - \frac{1}{2} E_v = \langle X_{u,u}, X_v \rangle = \Gamma_{11}^1 F + \Gamma_{11}^2 G$$

So we have $\begin{cases} L_1 = e \\ \Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \frac{1}{2} F_u - \frac{1}{2} E_v \end{cases}$ *i.e.*

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix} \tag{8.5}$$

So any $\Gamma_{ij}^k$ can be represented by $E, F, G$.

Now since $(X_{u,u})_v = (X_{v,v})_u$.

$$
\begin{aligned}
\frac{\partial}{\partial v} X_{u,u} &= ((\Gamma_{11}^1)_v X_u + \Gamma_{11}^1 X_{uv}) + ((\Gamma_{11}^2)_v X_v + \Gamma_{11}^2 X_{vv}) + ((e_1)_v N + e N_v) \\
&= (\Gamma_{11}^1)_v X_u + \Gamma_{11}^1((\Gamma_{12}^1) X_u + \Gamma_{12}^2 X_v + fN) \\
&+ (\Gamma_{11}^2)_v X_v + \Gamma_{11}^2(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + gN) \\
&+ e_v N + e(a_{12} X_u + a_{22} X_v)
\end{aligned} \tag{8.6}
$$

Similarly, one can calculate that

$$
\begin{aligned}
\frac{\partial}{\partial u} X_{u,v} &= (\Gamma_{12}^1)_u X_u + \Gamma_{12}^1(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + eN) \\
&+ (\Gamma_{12}^2)_u X_v + \Gamma_{12}^2(\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + fN) \\
&+ f_u N + f(a_{11} X_u + a_{21} X_v)
\end{aligned} \tag{8.7}
$$

The coefficients of $X_v$ is

$$
\Gamma_{11}^1 \Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2 \Gamma_{22}^2 + e a_{22} = \Gamma_{12}^1 \Gamma_{11}^2 + (\Gamma_{12}^2)_u + \Gamma_{12}^2 \Gamma_{12}^2 + f a_{21}
$$

where

$$
e a_{22} - f a_{21} = \frac{(efF - egE) - (feF - f^2 E)}{EG - F^2} = \frac{(f^2 - eg)E}{EG - F^2} = -KE
$$

So we obtain

$$
\frac{(\Gamma_{11}^1 \Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2 \Gamma_{22}^2) - (\Gamma_{12}^1 \Gamma_{11}^2 + (\Gamma_{12}^2)_u + \Gamma_{12}^2 \Gamma_{12}^2)}{E} = K \tag{8.8}
$$

Here we have proved that $K$ can be represented by $E, F, G$. $\square$

This is the **Gauss formula** we obtain

$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -EK \tag{8.9}$$

Similarly, one can prove for the coefficients of $X_u$

$$(\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = FK \tag{8.10}$$

It is (when $F \neq 0$) merely another form of the Gauss formula.

And the coefficients of $N$

$$e_v - f_u = e\Gamma^1_{12} - f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} \tag{8.11}$$

By applying the same process to $(x_{v,v})_u = (x_{u,v})_v$, we obtain the equation giving again the Gauss formula (8.9). Furthermore, we can obtain another equation

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} \tag{8.12}$$

(8.11) and (8.12) are called **Maindardi-Codazzi** equations

The Gauss formula and the Mainardi-Codazzi equations are known under the name of compatibility equations of the theory of surfaces.

**Theorem 8.15.** *If* $E, F, G, e, f, g$ *satisfies* (8.9) $\sim$ (8.12)*, then it uniquely determines a surface.*

### 8.3.1 Covariant derivative

For $S \hookrightarrow \mathbb{R}^3$ regular surface, $p \in S$, $X \in \Gamma(TS) = C^\infty(S, \mathbb{R}^3)$, $X : q \in S \to X_q \in T_qS \subset T_q\mathbb{R}^3 = \mathbb{R}^3$.

$y \in T_pS$ define covariant derivative of $X$ in the direction of $y$ *s.t.* $\nabla_y X =$

$Pj_{T_pS}(y(X))$ where $Pj_V$ is the orthogonal projection to $V$. So we define a map $X \in \Gamma(TX) \to \nabla_y X \in T_pS$.

Given vector fields $X, Y \in \Gamma(TS)$, define $\nabla_Y X \in \Gamma(TS)$ by

$$(\nabla_Y X)_p = \nabla_{Y_p} X \in T_pS, \ \forall p \in S \tag{8.13}$$

Then $\nabla : \Gamma(TS) \times \Gamma(TS) \to \Gamma(TS), (X, Y) \mapsto \nabla_Y X$.

**Lemma 8.16.**

① $\nabla$ *is bilinear.*

② $\nabla_{fY} X = f \cdot \nabla_Y X$

③ *(Leibniz rule)* $\nabla_Y (fX) = f\nabla_Y X + Y(f) \cdot X$.

④ *(compatibility with metric)* $Y(\langle X_1, X_2\rangle) = \langle \nabla_T X_1, X_2\rangle + \langle X_1, \nabla_Y X_2\rangle$

⑤ *(torsion free)* $\nabla_X Y - \nabla_X Y = [X, Y]$.

*Proof.* ① is straight forward.

②

$$\begin{aligned}
(\nabla_{fY} X)_p &= \nabla_{f(p)Y_p} X \\
&= pf(f(p)Y_p(X)) \\
&= f(p) \cdot Pj(Y_p(X)) \\
&= (f \cdot \nabla_Y X)_p
\end{aligned}$$

③

$$(\nabla_Y fX)_p = Pj(Y_p(fX))$$

$$= Pj(f(p)Y_p(X) + Y_p f X_p)$$
$$= f(p)Pj(Y_p(X)) + Y_p(f)X_p$$
$$= (f \cdot \nabla_Y X + Y(f)X)_p$$

④

$$\begin{aligned}(Y\langle X_1, X_2\rangle)_p &= \langle Y_p(X_1), X_{2,p}\rangle + \langle X_{1,p}, Y_p(X_2)\rangle \\ &= \left\langle \nabla_{Y_p} X_1, X_{2,p}\right\rangle + \left\langle X_{1,p}, \nabla_{Y_p} X_2\right\rangle \\ &= (\langle \nabla_Y X_1, X_2\rangle + \langle X_1, \nabla_Y X_2\rangle)_p\end{aligned}$$

⑤ Define $\mathrm{tor}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ Then

$$\begin{aligned}\mathrm{tor}(fX, Y) &= \nabla_{fX} Y - \nabla_Y fX - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - Y(f) \cdot X - f[X, Y] + Y(f)X \\ &= f\mathrm{tor}(X, Y)\end{aligned}$$

Take local chart $(U, x^1, x^2)$ of $S$. Suffices to check $\mathrm{tor}(X, Y) = 0$, $\forall X, Y \in \Gamma(TU)$.

For $X_1 = \partial x^1, X_2 = \partial x^2$, if we denote $X = f_1X_1 + f_2X_2$, $Y = g_1X_1 + g_2X_2$, then it suffices to check $\mathrm{tor}(X_1, X_2)$.

For the inclusion map $S \xhookrightarrow{\iota} \mathbb{R}^3$. $X_1 = \dfrac{\partial \iota}{\partial x^1} \in C^\infty(S, \mathbb{R}^3)$, $X_2 = \dfrac{\partial \iota}{\partial x^2} \in C^\infty(S, \mathbb{R}^3)$. Then

$$\nabla_{X_1} X_2 = Pj(\frac{\partial^2 \iota}{\partial x_1 \partial x_2})\nabla_{X_2} X_1$$

So $\mathrm{tor}(X_1, X_2) = 0$. □

**Definition 8.17.** A bilinear map $\nabla : \Gamma(TS) \times \Gamma(TS) \to \Gamma(TS)$ that satisfies ① ∼ ③ is called a **connection** on $TS$. A connection allows us to take covariant derivative of vector fields. A connection that satisfies ④,⑤ is called a **Levi-Civata connection**, denoted as $\nabla^{LC}$

**Theorem 8.18.** *$\nabla^{LC}$ is uniquely determined by the first fundamental form. i.e. $\nabla^{LC}$ is an intrinsic quantity. Equivalently, it is invariant under isometry.*

*Proof.* Take $(U, x^1, x^2)$, set $X_i = \dfrac{\partial}{\partial x^i} \in \Gamma(TU)$.

**Christoffel symbol** $\nabla_{X_i} X_j = \sum_{k=1,2} \Gamma^k_{i,j} X_k$, $\Gamma^k_{i,j} \in C^\infty(U, \mathbb{R})$. Then torsion free is equivalent to $\Gamma^k_{i,j} = \Gamma^k_{i,j}$.

Let $g_{i,j} = \langle X_i, X_j \rangle \in C^\infty(U, \mathbb{R})$ be the first fundamental form.

Then $\dfrac{\partial g_{11}}{\partial x^1} \overset{④}{=} 2 \langle \nabla_{X_1} X_1, X_1 \rangle \Rightarrow$

$$\Gamma^1_{11} g_{11} + \Gamma^2_{11} g_{12} = \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x^1} \tag{8.14}$$

$\dfrac{\partial g_{12}}{\partial X^1} \overset{④}{=} \langle \nabla_{X_1} X_1, X_2 \rangle + \langle X_1, \nabla_{X_1} X_2 \rangle = \langle \nabla_{X_1} X_1, X_2 \rangle + \langle X_1, \nabla_{X_2} X_1 \rangle = \langle \nabla_{X_1} X_1, X_2 \rangle + \dfrac{1}{2} \cdot \dfrac{\partial g_{11}}{\partial x^2}$. So

$$\langle \nabla_{X_1} X_1, X_2 \rangle = \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x^2} \tag{8.15}$$

*i.e.*

$$\Gamma^1_{11} g_{12} + \Gamma^2_{11} g_{22} = \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x^2} \tag{8.16}$$

So combined with (8.14) and (8.16)

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^1_{11} \end{pmatrix} = (g^{ij}) \cdot \begin{pmatrix} \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x^1} \\ \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2} \cdot \frac{\partial g_{11}}{\partial x^2} \end{pmatrix} \tag{8.17}$$

where $(g^{ij}) = (g_{ij})^{-1}$.

So $\Gamma^1_{11}$, $\Gamma^2_{12}$ is uniquely determined by $g_{ij}$. Similar for other $\Gamma^k_{ij}$. $\square$

A natural question is to compute $\nabla_{X_1}\nabla_{X_2}-\nabla_{X_2}\nabla_{X_1}$, or $\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$ more generally.

Define $R(X,Y,Z)\in\Gamma(TS)$ by $\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$.

**Proposition 8.19.** $\forall f,h,\rho\in C^\infty(S,\mathbb{R})$, $R(fX,hZ,\rho Z)=fh\rho R(X,Y,Z)$.

**Corollary 8.20.** *$R(X,Y,Z)_p$ only depends on $X_p,Y_p,Z_p$ i.e. $R\in\Gamma(\mathrm{Hom}(TS\otimes TS\otimes TS,TS))=\Gamma(T^*S\otimes T^*S\otimes T^*S\otimes TS)$ is a tensor field.*

*Proof of corollary.* Left as exercise. □

*Proof of Proposition.* Consider the case $f\equiv h\equiv 1$.

$$\begin{aligned}
R(X,Y,\rho Z) &= \nabla_X\nabla_Y\rho Z-\nabla_Y\nabla_X\rho Z-\nabla_{[X,Y]}\rho Z\\
&= \nabla_X(\rho\nabla_Y Z+Y(\rho)Z)-\nabla_Y(\rho\nabla_X Z+X(\rho)Z)-\rho\nabla_{[X,Y]}Z-[X,Y](\rho)Z\\
&= \rho\nabla_X\nabla_Y Z+X(\rho)\nabla_Y Z+Y(\rho)\nabla_X Z+XY(\rho)Z\\
&- \rho\nabla_Y\nabla_X Z-Y(\rho)\nabla_X Z-X(\rho)\nabla_Y Z-YX(\rho)Z\\
&- \rho\nabla_{[X,Y]}Z-[X,Y](\rho)(Z)\\
&= \rho R(X,Y,Z)
\end{aligned}$$

□

$R(X,Y,Z)$ is an intrinsic quantity. There is a relation with Gauss curvature $K$.

**Theorem 8.21.** *For any orthogonormal basis $V_1,V_2$ of $T_pS$, $R(V_1,V_2,V_1)=-K(p)V_2$.*

**Corollary 8.22** (Gauss Theorem EGREGIUM)**.** *$K$ is an intrinsic quantity.*

*Proof.* Take local chart $(U,x^1,x^2)$ *s.t.* $V_1=X_{1,p}$, $V_2=X_{2,p}$. $S\overset{\rho}{\hookrightarrow}\mathbb{R}^3$. Then $X_i=\dfrac{\partial\rho}{\partial x^i}\in C^\infty(S,\mathbb{R}^3)$. For any $i,j$, define $X_{j,k}=\dfrac{\partial^2\rho}{\partial x^j\partial x^k}$.

$\forall p\in U$, $(X_{1,p},X_{2,p},N_p)$ is a basis for $T_p\mathbb{R}^3$, where $N:S\to S^2\subset\mathbb{R}^3$ is the Gauss map. Then

$$X_{11} = X_1(X_1) = \nabla_{X_1} X_1 + L_1 N$$

$$X_{12} = X_2(X_1) = \nabla_{X_2} X_1 + L_2 N$$

$$X_{21} = \nabla_{X_1} X_2 + L_2 N$$

$$X_{22} = \nabla_{X_2} X_2 + L_3 N$$

where $L_1, L_2, L_3 \in C^\infty(U, \mathbb{R})$. The second fundamental form is given by matrix $\begin{pmatrix} L_1 & L_2 \\ L_2 & L_3 \end{pmatrix}$.

$$\begin{aligned} R(X_1, X_2, X_1) &= \nabla_{X_1} \nabla_{X_2} X_1 - \nabla_{X_2} \nabla_{X_1} X_1 \\ &= \nabla_{X_1}(X_{12} - L_2 N) - \nabla_{X_2}(X_{11} - L_1 N) \\ &= Pj(X_{112} - X_1(L_2 N) - X_{211} + X_2(L_1 N)) \\ &= Pj(L_1 X_2(N) - L_2 X_1(N)) \\ &= Pj(L_1 N_2 - L_2 N_1) \end{aligned}$$

As we proved before, (see Remark 8.5)

$$\begin{cases} \langle X_1, N_1 \rangle = -L_1 \\ \langle X_1, N_2 \rangle = -L_2 \\ \langle X_2, N_1 \rangle = -L_2 \\ \langle X_2, N_2 \rangle = -L_3 \end{cases} \tag{8.18}$$

and $\langle N, N_1\rangle = 0$ since $\langle N, N\rangle \equiv 1$. $X_1, X_2, N$ orthogonormal $\Rightarrow$

$$N_1 = -L_1 X_1 - L_2 X_2,\ N_2 = -L_2 X_1 - L_3 X_2$$

Then $R(X_1, X_2, X_1) = Pj(L_1 N_2 - L_2 N_1) = Pj(L_2^2 X_2 - L_1 L_3 X_2) = -Pj(K X_2) = -K$. □

## 8.4 Parallel transport

For $\gamma : I \to S$ curve on $S$, a vector field $W$ along $\gamma$ is an assignment $t \in I \rightsquigarrow W_t \in T_{\gamma(t)}S \subset \mathbb{R}^3$. We can view $W$ as $W \in C^\infty(I, \mathbb{R}^3)$. *i.e.*

$$\{\text{vector fields over } \gamma\} = \{W : I \to \mathbb{R}^3 | W(t) \in T_{\gamma(t)}S, \forall t \in I\} = \Gamma(\gamma^* TS)$$

where $\gamma^* TS$ is the pullback of $TS$.

Define **covariant derivative** (along $\gamma$)

$$\nabla_{\gamma'(t)}(W) = Pj_{T_{\gamma(t)}S}\left(\frac{\mathrm{d}W}{\mathrm{d}t}\right) \in \Gamma(\gamma^* TS)$$

which is another vector field over $\gamma$.

**Example 8.23.**

(1) $\gamma'(t) \in \Gamma(\gamma^*(TS))$.

(2) $\gamma$ parametrized by arc length, $\forall t$, $\exists$ unique $N^\gamma_{\gamma(t)} \in T_{\gamma(t)}S$ *s.t.* $(\gamma'(t), N^\gamma_{\gamma(t)})$ is an oriented orthogonormal basis for $T_{\gamma(t)}S$. So we can define the normal ve tor field of $\gamma$ as $N^\gamma : t \mapsto N^\gamma_{\gamma(t)}$.

(3) Given $X \in \Gamma(TS)$, $X|_\gamma \in \Gamma(\gamma^* TS), t \mapsto X_{\gamma(t)}$, then we have

$$\nabla_{\gamma'(t)}(X|_\gamma) = \nabla_{\gamma'(t)} X$$

So the notation is compatible.

Given $W \in \Gamma(\gamma^*(TS))$, say $W$ is **parallel** if $\nabla_{\gamma'(t)} W = 0$, $\forall t \in I$.

We say $\gamma$ is a **geodesic** if $\gamma'(t)$ is parallel, *i.e.* $\nabla_{\gamma'(t)}\gamma'(t) = 0$, which generalizes the straight line in $\mathbb{R}^2$.

For $\gamma : I \to S \hookrightarrow \mathbb{R}^3$ parametrized by arc length. Then $|\gamma'(t)| = 1 \Rightarrow \gamma''(t) \perp \gamma'(t) \Rightarrow \gamma''(t) = K_g \cdot N^\gamma_{\gamma(t)} + K_n + N^S_{\gamma(t)}$ where $K_g$ is called the **geodesic curvature** and $K_n$ is the **normal curvature**.

$K(\gamma)^2 = K_g^2 + K_n^2$ is called the **curvature** of $\gamma$ as curve in $\mathbb{R}^3$.

$\nabla_{\gamma'(t)}\gamma'(t) = K_g N^\gamma_{\gamma(t)}$ so $\gamma$ is geodesic if and only if $K_g = 0$.

**Example 8.24.** $S = S^2 = \{x^2 + y^2 + z^2 = 1\}$, $\gamma : \mathbb{R} \to S$, $\gamma(t) = (\cos t, \sin t, 0)$. Then $\gamma'(t) = (-\sin t, \cos t, 0)$, $N^\gamma_{\gamma(t)} = (0, 0, 1)$, $N^S_{\gamma(t)} = (\cos t, \sin t, 0)$. $\gamma''(t) = (-\cos t, -\sin t, 0) = -N^S_{\gamma(t)} \Rightarrow K_g = 0$, $K_n = -1 \Rightarrow \gamma$ is a geodesic.

Indeed, there is a fact about geodesic in $S^2$

**Fact 8.25.** *$\gamma : I \to S^2$ is a geodesic if and only if $\gamma$ moves along a big circle in a constant speed.*

**Fact 8.26.** *$\gamma : I \to S$ is a geodesic if and only if $\forall t_0 \in I, \exists \varepsilon > 0$ s.t. $\forall t_1 \in I, |t_1 - t_0| < \varepsilon$, we have $l(\gamma|_{[t_0,t_1]}) = \mathrm{d}(\gamma(t_0), \gamma(t_1))$, where*

$$l(\eta) = \int_I \eta'(t)\mathrm{d}t,\ \mathrm{d}(p, q) = \min\{l(\eta) | \eta : [0, 1] \to S\ \eta(0) = p, \eta(1) = q\}$$

*i.e. Geodesic is the shortest path between two points locally.*

## 8.5 Gauss-Bonnet Theorem

**Theorem 8.27** (Gauss-Bonnet Theorem)**.** *$S$ closed and oriented surface. Then*

$$\int_S K\mathrm{dVol} = 2\pi\chi(S) \tag{8.19}$$

There is a generalization for a famous result: The sum of the outer angles of the polygon is $2\pi$.

For $\gamma : I = [0, T] \to S$.

Say $\gamma$ is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ for $t_1 \neq t_2$ (except $t_1 0, t_2 = T$)

Say $\gamma$ is **closed** if $\gamma(0) = \gamma(T)$.

Say $\gamma$ is piece $C^1$ if $\exists\ 0 = t_0 < t_1 < \cdots < t_n = T \quad s.t.\ \gamma_i = \gamma|_{[t_i,t_{i+1}]}$ is $C^1$.

For each $i$, we have

$$\gamma'_-(t_i) := \lim_{t\to t_i^-} \frac{\gamma(t) - \gamma(t_i)}{t - t_i}, \gamma'_+(t_i) = \lim_{t\to t_i^+} \frac{\gamma(t) - \gamma(t_i)}{t - t_i}$$

They are both in $T_{\gamma(t)}S$. $\theta_i$ is defined as the angle from $\gamma'_-(t_i)$ to $\gamma'_+(t_i)$, $\theta_i \in [-\pi, \pi]$.

$\gamma : I \to S \hookrightarrow \mathbb{R}^3$, $\gamma(I) \subset U \cong \mathbb{R}^2$, $U \xrightarrow[\cong]{(x^1,x^2)} \mathbb{R}^2$ oriented local chart. $X_1 = \frac{\partial}{\partial x^1}, X_2 = \frac{\partial}{\partial x^2} \in \Gamma(TU) \overset{\text{GS}}{\rightsquigarrow} e_1, e_2 \in \Gamma(TU)$ such that $\langle e_i, e_j\rangle = \delta_{ij}$.

Assume $\gamma$ parametrized by arc length $t$, *i.e.* $\gamma'(t) = 1 \Rightarrow \exists\varphi : I \to \mathbb{R}$ *s.t.* $\gamma'(t) = \cos\varphi(t) \cdot e_{1,\gamma(t)} + \sin\varphi(t) e_{2,\gamma(t)}$.

Define the rotation number of $\gamma$, $\mathrm{rot}(\gamma) = \varphi(t_1) - \varphi(t_0)$ for $I = [t_0, t_1]$.

**Theorem 8.28.** *Let $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ be a piecewise $C^1$ simple closed curve whose image is contained in $U \subset S$, $\gamma : I \to U \subset S \hookrightarrow \mathbb{R}^3$. Then*

$$\sum_{i=1}^n \mathrm{rot}(\gamma_i) + \sum_{i=1}^n \theta_i = 2\pi$$

*Proof.* When $S = \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ is the standard embedding, see Do Garmo. Here is a simplified proof

For $\triangle = \{(x,y) \in \mathbb{R}^2 | 0 \leqslant x \leqslant y \leqslant 1\}$, $f : \triangle \to S^1$

$$f(x,y) = \begin{cases} \frac{\gamma(x)-\gamma(y)}{|\gamma(x)-\gamma(y)|}, & x \neq y,\ (x,y) \neq (0,1) \\ \frac{\gamma'(x)}{|\gamma'(x)|}, & x = y \\ -\gamma'(0) & (x,y) = (0,1) \end{cases} \tag{8.20}$$

Then $f|_C, f|_{A\cup B}$ are loops in $S^1$, where $C$ is the hypotenuse and $A, B$ are the legs of this triangle.

Observe that $2\pi \cdot \deg(f|_C) = \operatorname{rot}(\gamma) = 2\pi \deg(f|_{A\cup B})$, $f(0,y) = \dfrac{\gamma(y)-\gamma(0)}{|\gamma(t)-\gamma(0)|} = -f(s,1)$.

Since $\exists L \subset \mathbb{R}^2$, $L$ tangent to $\gamma$ at $\gamma(0)$, $\gamma$ fall on one side of $L \Rightarrow f|_A$ is not surjective. By considering fundamental group, we can prove $\deg(f|_{A\cup B}) = 1$. Thus, $\deg(f|_C) = 1$, which is what we need. □

Here are some notations we use below: $(U, x^1, x^2)$ chart of $S$, $X_1, X_2, e_1, e_2$ defined above.

**Proposition 8.29.** *$\exists \alpha \in \Omega^1(U)$ s.t. $\nabla_V e_1 = \alpha(V)e_2$, $\nabla_V e_2 = -\alpha(V)e_1$, $\forall V \in \Gamma(TU)$. Furthermore, $K\mathrm{dVol} = -\mathrm{d}\alpha$.*

*Proof.*

$$\begin{cases} \langle e_2, e_2\rangle = 1 \\ \langle e_1, e_1\rangle = 1 \\ \langle e_1, e_2\rangle = 0 \end{cases} \Rightarrow \begin{cases} \langle \nabla_V e_1, e_1\rangle = 0 \\ \langle \nabla_V e_2, e_2\rangle = 0 \\ \langle \nabla_V e_1, e_2\rangle = -\langle e_1, \nabla_V e_2\rangle \end{cases} \Rightarrow \begin{cases} \nabla_V e_1 = \tau(V)e_2 \\ \nabla_V e_2 = -\tau(V)e_1 \end{cases}$$

for some $\tau : \Gamma(TU) \to C^\infty(U)$.

$\nabla_{fV}e_1 = f\nabla_V e_1 \Rightarrow \tau(fV) = f\tau(V)$. So $\tau$ is a (0,1)-tensor, *i.e.* $\tau \in \Gamma(\mathrm{Hom}(TU, \mathbb{R}))$. $\Rightarrow \exists\, \alpha \in \Omega^1(U)$ *s.t.* $\tau(V) = \alpha(V)$.

Still need to prove $(K\mathrm{d}\alpha)_p = (-\mathrm{d}\alpha)_p$. $\forall p \in U$. We may assume $X_{1,p} = e_{1,p}$ and $X_{2,p} = e_{2,p}$ that are orthogonormal.

By theorem 8.21, $R(X_{1,p}, X_{2,p}, e_{1,p}) = -K(p)e_{2,p}$. *i.e.*

$$\begin{aligned} -K(p)e_{2,p} &= R(X_{1,p}, X_{2,p}, e_{1,p}) \\ &= (\nabla_{X_1}\nabla_{X_2}e_1 - \nabla_{X_2}\nabla_{X_1}e_1)_p \\ &= (\nabla_{X_1}\alpha(X_2)e_2 - \nabla_{X_2}\alpha(X_1)e_2)_p \\ &= (X_1\alpha(X_2) - X_2\alpha(X_1))_p \cdot e_{2,p} \\ &= \mathrm{d}\alpha(X_1, X_2)_p e_{2,p} \end{aligned}$$

So $(\mathrm{d}\alpha)_p = -K(p)e_1^* \wedge e_2^* = -K(p)(\mathrm{dVol})_p$ □

**Theorem 8.30** (Local Gauss-Bonnet Theorem). *$\gamma : I \to U \subset S \hookrightarrow \mathbb{R}^3$, $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$ simple closed and piecewise $C^1$ curve. $\gamma$ bounds a region $R \subset U$, oriented $\gamma$ as $\partial R$. Then*

$$\sum_{i=1}^n \int_{\gamma_i} K_g \mathrm{d}S + \sum_{i=1}^n \theta_i + \int_R K\mathrm{dVol} = 2\pi$$

*Proof.* For $\gamma$ parametrized by arc length, let

$$\gamma_i'(t) = \cos\varphi_i(t)e_1 + \sin\varphi_i(t)e_2,\ N^{\gamma_i}_{\gamma_i(t)} = -\sin\varphi_i(t)e_1 + \cos\varphi_i(t)e_2$$

Then

$$\begin{aligned} \nabla_{\gamma'(t)}\gamma'(t) &= \cos\varphi_i(t)\nabla_{\gamma'(t)}e_1 + \sin\varphi_i(t)\nabla_{\gamma'(t)}e_2 \\ &= \alpha(\gamma_i'(t))N^{\gamma}_{\gamma'(t)} + \varphi_i'(t) \cdot N^{\gamma}_{\gamma(t)} \\ &= (\alpha(\gamma_i'(t)) + \varphi_i'(t))N^{\gamma}_{\gamma(t)} \end{aligned}$$

$$K_g = \alpha(\gamma_i'(t)) + \varphi_i'(t) \Rightarrow \int_{\gamma_i} K_g \mathrm{d}S = \int_{\gamma_i} \alpha + \mathrm{rot}(\gamma_i) \Rightarrow$$

$$\begin{aligned}
\sum_{i=1}^{n} \int_{\gamma_i} K_g \mathrm{d}S + \sum_{i=1}^{n} \theta_i &= \sum_{i=1}^{n} \int_{\gamma_i} \alpha + \sum_{i=1}^{n} \mathrm{rot}(\gamma_i) + \sum_{i=1}^{n} \theta_i \\
&= \int_{\gamma} \alpha + 2\pi \\
&= \int_{R} \mathrm{d}\alpha + 2\pi \\
&= -\int_{R} K \mathrm{dVol} + 2\pi
\end{aligned}$$

□

To prove the general Gauss-Bonnet theorem, only need to use triangulation of $S$.

# Index

# List of Theorems